AST4320: LECTURE 6

M. DIJKSTRA

1. Gaussian Random Fields & The Press Schechter Formalism

1.1. **Support for Previous Lecture.** The plot shown in the slides where the cooling time was compared to the collapse time contains lines of constant mass. These masses are virial masses:

$$M = \frac{4}{3}\pi\rho R_{\rm vir}^3. \tag{1}$$

We defined the circular velocity $v_{\rm circ}^2 = \frac{GM}{R_{\rm fir}}$, and $kT_{\rm fir} = \frac{1}{2}m_{\rm p}v_{\rm circ}^2$ (where $m_{\rm p}$ denotes proton mass). We can combine these last two expressions to get $R_{\rm vir} = \frac{GMm_p}{2kT_{\rm vir}}$. In other words, the virial mass $M \propto \rho R_{\rm vir}^3 \propto \rho\frac{M^3}{T_{\rm vir}^3}$. This proportionality can only be obeyed if $\rho\frac{M^2}{T_{\rm vir}^3}$ =constant. This explains the why lines of constant mass lie in this diagram the way they do: for example, for a constant $T_{\rm vir}$ we have ρM^2 =constant, i.e. $M \propto \rho^{-1/2}$ as is the case in the Figure.

1.2. **Introduction.** Temperature fluctuations in the cosmic microwave background are Gaussian. In the standard cosmological model these T-fluctuations are sourced by density fluctuations, which must also be Gaussian. Gaussian random fields play a key role in theory of structure formation, and we summarise some properties here. First, we consider a single-variate Gaussian distribution. This has many of the features which return in the more general (and cosmologically relevant) multivariate Gaussian distribution. We then summarise properties of a Multi-variate Gaussian. This is followed by discussing properties of Gaussian random fields in Fourier space. Here, we introduce the power spectrum, which plays a central role in structure formation theories. This discussion follows that in the lecture of Eiichiro Komatsu: `http://www.mpa-garching.mpg.de/~komatsu/cmb/lecture_NG_iucaa_2011.pdf`.

1.3. **Single-Variate Gaussian Random Fields.** We have a single variable x drawn from the probability distribution function (PDF) $p(x)$, which is normalised to $\int_{-\infty}^{\infty} dx\, p(x) = 1$. If x obeys Gaussian statistics then

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{x^2}{2\sigma^2}\Big) \tag{2}$$

, where σ^2 denotes the variance. The PDF contains the full information of the field. Various useful statistical quantities of the field are 'moments' of the field. The first four are:

$$\langle x\rangle = \int_{-\infty}^{\infty} dx\ xp(x) = 0 \tag{3}$$
$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx\ x^2 p(x) = \sigma^2$$
$$\langle x^3\rangle = \int_{-\infty}^{\infty} dx\ x^3 p(x) = 0$$
$$\langle x^4\rangle = \int_{-\infty}^{\infty} dx\ x^4 p(x) = 3\sigma^4.$$

These 4 moments indicate that the field has zero mean (first moment), variance σ^2 (second moment), zero skewness (third moment), zero kurtosis (fourth moment: the kurtosis $\kappa_4 \equiv \langle x^{\rangle} - 3\langle x^2\rangle^2 = 0$). For a single-variate Gaussian random field, all odd moments vanish, all even moments are given in terms of σ^{2n}.

1.4. **Multi-Variate Gaussian Random Fields.** The general expression for a multi-variate (here there are N variables) Gaussian-PDF is given by

$$p(x_1, x_2, ..., x_{\rm N}) = \frac{1}{(2\pi)^{N/2}|\xi|}\exp\Big(- 0.5\sum_{ij} x_i(\xi_{ij}^{-1})x_j\Big), \tag{4}$$

where ξ_{ij} denotes the covariance matrix or the two-point correlation function. Moments of this PDF are

$$\langle x_i\rangle = \underbrace{\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 ... \int_{-\infty}^{\infty} dx_N}_{\int d^N x}\ x_i\ p(x_1, x_2, ..., x_{\rm N}) = 0 \tag{5}$$
$$\langle x_i x_j\rangle = \int d^N x\ x_i x_j\ p(x_1, x_2, ..., x_{\rm N}) = \xi_{ij}$$
$$\langle x_i x_j x_l\rangle = \int d^N x\ x_i x_j x_l\ p(x_1, x_2, ..., x_{\rm N}) = 0$$
$$\langle x_i x_j x_l x_m\rangle = \int d^N x\ x_i x_j x_l x_m\ p(x_1, x_2, ..., x_{\rm N}) = \xi_{ij}\xi_{lm} + \xi_{il}\xi_{jm} + \xi_{im}\xi_{jl}.$$

This is very similar to the single variate case: odd moments vanish. Even moments are given in terms of ξ. Note that $\langle x_i x_i\rangle = \xi_{ii} = \sigma^2$. Moreover, $\langle x_i x_i x_i x_i\rangle = 3\sigma^4$ exactly as in the single-variate case.

Next, we will use that the Universe is isotropic (invariant under rotation) and homogeneous (invariant under translation). Let x now be a variable that depends on position $\mathbf{q}$. The two-point correlation function ξ_{ij} is

$$\xi_{ij} = \langle x(\mathbf{q}_i)x(\mathbf{q}_j)\rangle = \langle x(\mathbf{q}_i)x(\mathbf{q}_i + \mathbf{r}_{ij})\rangle, \tag{6}$$

where $\mathbf{r}_{ij}$ is a vector that connects the two points. Homogeneity (translational invariance) requires that ξ_{ij} is a function of $\mathbf{r}_{ij}$) alone, and not of $\mathbf{q}_i$. Homogeneity and isotropy then requires that ξ_{ij} is a function of $|\mathbf{r}_{ij}|$ alone.

1.5. **Multi-Variate Gaussian Random Fields in Fourier Space.** Working in Fourier space has its advantages. The Fourier transform of $x(\mathbf{q})$ is given by

$$\tilde{x}(\mathbf{k}) = \int d^3\mathbf{q}\exp(-i\mathbf{k}\cdot\mathbf{q})x(\mathbf{q}). \tag{7}$$

The PDF of the Fourier components is given by

$$p(\hat{x}(\mathbf{k}_1), \hat{x}(\mathbf{k}_2), ..., \hat{x}(\mathbf{k}_N)) = \frac{1}{(2\pi)^{N/2}|C|^{1/2}} \exp\Big(-0.5\sum_{ij}\tilde{x}(\mathbf{k}_i)(C^{-1}_{ij})\tilde{x}^*(\mathbf{k}_j)\Big), \tag{8}$$

where $\tilde{x}^*$ denotes the complex conjugate of $\tilde{x}$. Now, similar to the previous cases

$$\langle\tilde{x}(\mathbf{k}_i)\tilde{x}^*(\mathbf{k}_j)\rangle = C_{ij}. \tag{9}$$

Assuming spatial homogeneity, we can show that (**Assigment 3**)

$$\begin{aligned} C_{ij} &= (2\pi)^3\delta_{\mathrm{D}}(\mathbf{k}_i - \mathbf{k}_j)P(\mathbf{k}_j) \\ P(\mathbf{k}_j) &= \int d^3\mathbf{r}\exp(-i\mathbf{k}\cdot\mathbf{r})\xi(\mathbf{r}). \end{aligned} \tag{10}$$

Here, $P(\mathbf{k}_j)$ is known as the 'power spectrum'. Assuming additional isotropy, we have that$P(\mathbf{k}_j) = P(|\mathbf{k}_j|)$. Using these expressions can 'simplify'[1] the PDF of the Fourier components:

$$p(\hat{x}(\mathbf{k}_1), \hat{x}(\mathbf{k}_2), ..., \hat{x}(\mathbf{k}_N)) = \frac{1}{(2\pi)^{N/2}|\Pi_i P(|\mathbf{k}_i)|)|^{1/2}} \exp\Big(-0.5\sum_i\frac{|\tilde{x}(\mathbf{k}_i)|^2}{P(|\mathbf{k}_i|)}\Big). \tag{11}$$

The reason that we call this simplified is that there is no $N \times N$ matrix anymore. Instead, all information is now encoded in the N-dimensional vector $P(\mathbf{k}_i)$, the power spectrum. This power spectrum is a popular way to describe the information of a (Gaussian) density field.

[1]It still looks a bit ugly.

1.6. **Press-Schechter Formalism.** We are interested in the probability that an atom at position $\mathbf{q}$ is part of a collapsed object with mass $> M$. This probability is denoted with $P(> M)$. The Press-Schechter *ansatz* is that

$$P(> M) = P(\delta > \delta_{\rm crit}|M) \tag{12}$$

where $P(\delta|M)$ denotes the PDF of the density field smoothed on some scale R - which corresponds to a mass scale M. The idea is that if $\delta > \delta_{\rm crit}$ on some scale M, then $\delta = \delta_{\rm crit}$ on some larger mass-scale M' and the atom would be part of the larger collapsed mass M'.

The temperature fluctuations of the Cosmic Microwave Background tell us that the mass-density field was Gaussian at early times. The mass-density field smoothed over some mass-scale M is also a Gaussian random field. We can the compute the probability

$$P(\delta > \delta_{\rm crit}|M) = \frac{1}{\sigma(M)\sqrt{2\pi}} \int_{\delta_{\rm crit}}^{\infty} d\delta' \, \exp\Big(- \frac{\delta'^2}{2\sigma^2(M)}\Big), \tag{13}$$

where $\sigma^2(M)$ denotes the variance of the mass-density field smoothed over scale M. This integral can be re-expressed as an error-function[2]:

$$P(> M) = P(\delta > \delta_{\rm crit}|M) = \frac{1}{2}\Big[1 - {\rm erf}\Big(\frac{\nu}{\sqrt{2}}\Big)\Big], \tag{14}$$

where $\nu \equiv \frac{\delta_{\rm crit}}{\sigma(M)}$. This idea has problem with it, if we take $M \to 0$, then $P(> M) = 0.5$. This is because for a Gaussian random field, half the density fluctuations are under dense. In non-linear evolution, matter from the under dense regions will fall onto the massive objects (i.e. empty regions get emptier, dense regions become denser). This was a well known problem as soon as the theory was proposed. It was fixed my multiplying by a factor[3] of 2. We will do that as well.

Because $P(> M)$ denotes the fraction of atoms locked up in an object of mass $> M$, $\frac{\partial P}{\partial M} dM$ denotes the mass fraction of the Universe locked up into collapsed dark matter halos of masses in the range $M \pm dM/2$. This fraction can be related to the dark matter halo mass function, $n(M)dM$, which denotes the number density of dark matter halos with masses in the range $M \pm dM/2$. To see this connection, we

- Note that $Mn(M)dM$ denotes the mass density in dark matter halos with masses in the range $M \pm dM/2$.
- Note that $Mn(M)dM/\rho_{\rm m}$ denotes the fraction of the mass density (i.e. mass fraction) in dark matter halos with masses in the range $M \pm dM/2$.

[2]The error function is define as ${\rm erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^2 \exp(-t^2) dt$.

[3]This factor of 2 follows naturally from extension of the theory, and relates to the fact that the standard theory does not take into account at all that collapsed objects of some smaller mass M'' can end up in a more massive object of mass M. This problem is known as the 'cloud-in-cloud' problem.

We therefore have (where we have include Press & Schechters infamous factor of 2)

$$Mn(M)dM/\rho_{\rm m} = \frac{\partial P}{\partial M}dM \Rightarrow \tag{15}$$

$$n(M) = \frac{\rho_{\rm m}}{M}\frac{\partial P}{\partial M} = \frac{\rho_{\rm m}}{M}\frac{\partial}{\partial M}\Big[1 - {\rm erf}\Big(\frac{\nu}{\sqrt{2}}\Big)\Big] =$$

$$= \frac{\rho_{\rm m}}{M}\frac{\sqrt{2}}{\sqrt{2}}\frac{\partial}{\partial \nu}\Big[1 - {\rm erf}\Big(\frac{\nu}{\sqrt{2}}\Big)\Big]\frac{\partial \nu}{\partial M} = \frac{\rho_{\rm m}}{M}\frac{1}{\sqrt{2}}\frac{\partial}{\partial x}\Big[1 - {\rm erf}(x)\Big]\frac{\partial \nu}{\partial M} =$$

$$= -\frac{\rho_{\rm m}}{M}\frac{1}{\sqrt{2}}\frac{2}{\sqrt{\pi}}\exp(-x^2)\frac{\delta_{\rm crit}}{\sigma^2(M)}\frac{\partial \sigma}{\partial M}$$

What remains to be done is to characterise our Gaussian random field, which we do with the power spectrum $P(k)$ which enters our expression through its influence on $\sigma(M)$. We will sketch this relation next.

1.7. **Relation Between $P(k)$ and $\sigma(M)$.** We know that $\xi(\mathbf{r})$ and $P(\mathbf{k})$ are Fourier transforms of each other:

$$\xi(r) = \int_0^\infty \frac{d^3\mathbf{k}}{(2\pi)^3}P(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}) \tag{16}$$

We also know from our analysis on Gaussian random fields that $\sigma^2 = \xi(\mathbf{r}=0)$. Therefore,

$$\sigma^2 = \int_0^\infty \frac{d^3\mathbf{k}}{(2\pi)^3}P(\mathbf{k}) \underset{\rm isotropy}{=} \frac{1}{2\pi^2}\int dk\ k^2 P(k) \propto \int dk\ k^{n+2} \tag{17}$$

where we assumed that $P(k) = Ak^n$. The value σ^2 is variance in mass at one location. We are interested in the variance $\sigma^2(M)$, smoothed over some mass scale M, which corresponds to a spatial scale R. Smoothing the density field over some scale R corresponds to removing all fluctuations on scales smaller than R, or in other words, suppressing the power spectrum on wave numbers larger than $k = 2\pi/R$. We therefore have

$$\sigma^2(M) = \sigma^2(R) = \int_0^{2\pi/R}\frac{d^3\mathbf{k}}{(2\pi)^3}P(\mathbf{k}) \propto \int^{R^{-1}} dk\ k^{n+2} \propto R^{(-n-3)} \propto M^{(-n-3)/3}. \tag{18}$$

The main point is that we can compute $\sigma(M)$ (and hence predict the dark matter halo mass function) once we specified $P(k)$. The Press-Schechter Theory does a remarkable job at reproducing simulated mass functions that properly account for all gravitational effects.

1.8. **Useful Reading.** Useful reading material includes:

- Lecture on Gaussian random fields was taken from lecture given by Eiichiro Komatsu `http://www.mpa-garching.mpg.de/~komatsu/cmb/lecture_NG_iucaa_2011.pdf`. This discussion also contains extensive discussion on non-Gaussianity.
- Press-Schechter Theory: Textbooks by J. Peacock (Chapter 17), and M. Longair (see index) have descriptions on which this lecture was based.