UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220 — Cosmology II

Date: Wednesday, June 12th, 2024

Time: 15:00 – 19:00

The exam set consists of 11 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Each problem counts for 20% of the final score. Note that the problems 2-5 are ordered according to the order the topic was covered in the lectures and not by difficulty.

Problem 1 – Background questions (20 points)

Answer each question with at most three or four sentences.

a) Explain qualitatively how inflation gave rise to the fluctuations that seeded the subsequent growth of structure in the universe?

Quantum fluctuations in the inflaton field makes inflation last slightly longer or slightly shorter at different points in space. This leads to classical perturbations in the energy density and pressure which serve as initial conditions for the subsequent growth of structure in the universe.

b) What is the equation of state for a perfect fluid? Mathematically what does it mean that the Universe is accelerating and what condition must the equation of state of a fluid responsible for an accelerated expansion of the Universe satisfy?

The equation of state is the ratio $w = p/\rho$ with p being pressure and ρ being density. For cold matter $w = 0$ and for a radiation gas $w = 1/3$. Accelerating means $\ddot{a} > 0$. From the second Friedmann equation $\ddot{a} \propto -(\rho + 3P) \propto -(1 + 3w)$ and we see that we need $w < -1/3$ in order to have acceleration.

c) What are the main advantages of line of sight integration over the traditional Boltzmann hierarchy approach?

The main advantage is computational speed. By formally integrating the Boltzmann equations before expanding into multipole moments, as opposed to expanding before integrating, only only has to solve for 6 photon multipole moments, rather than 1000s of moments, even when calculating the full CMB spectrum to high ℓ's. A second advantage is a clearer physical interpretation of the various effects that impact the spectrum, such as Sachs-Wolfe, Doppler, Integrated Sachs-Wolfe etc.

d) What is the visibility function and what is the physical interpretation of this? Why it is practically zero at very early times?

The visibility function is defined as $g = \frac{d}{d\eta}(e^{-\tau})$ and satisfies $\int g d\eta = 1$. It represents the probability density for a photon last scattering at time η and has a peak at the last scattering surface. It is zero at very early times because its very improbable that a photon made it from the very early Universe till today without scattering at all (photons scattered many times in the tight coupling regime). At late times the Universe gets reionized, this causes some of the photons free-streaming towards us from the LSS to scatter so we get another smaller (and broader) peak in the visibility function.

e) What do the Fermi-Dirac and Bose-Einstein distributions describe? What does the Maxwell distribution describe?

The Fermi-Dirac distribution describes the distribution function for fermions (spin-half particles, such as electrons, protons etc.), while the Bose-Einstein describes the distribution function for bosons (spin-integer particles, such as photons, gravitons etc.) The Maxwell distribution is the low-temperature limiting case of both distributions.

f) Why do we solve the Einstein-Boltzmann equations in Fourier-space instead of real-space?

Since the first order equations are linear, the different Fourier modes decouple, so you get a separate equation set for each mode k. This turns the set of coupled partial differential equations into decoupled sets of ordinary differential equations, which are much easier to solve.

g) What is the relationship between conformal time η and cosmic time t? Give a physical interpretation of conformal time.

The relation is given by $dt = a d\eta$. This makes the metric equal to Minkowski times a^2 (i.e. the metric is then a so-called conformal transformation of Minkowski). The conformal time η (times c) represents the particle horizon at any given time.

h) The Einstein equation reads

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

What does the left- and right-hand sides of this equation describe, respectively? Why is it a good approximation to linearize this equation around a FRW metric when studying large scale structure formation?

The left-hand side of the equation is the Einstein tensor, which depends only on the metric, and therefore describes geometry of space. The right-hand side is the energy-momentum tensor, and describes the energy-matter content of space. For large scale structure the perturbations to the metric are always small (e.g. a galaxy cluster only has $\Phi \sim 10^{-5}$) so its a good approximation to linearize.

Problem 2 – The Saha Equation (20 points)

The Saha equation plays a central role when calculating the CMB power spectrum. In the following, we will derive one form of this equation suitable for this purpose. For a gas consisting of photons, protons and electrons in thermodynamic equilibrium, one can show that the following relation, the Saha equation, holds

$$\frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} = \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}},$$

where n_X is the density of species X, and

$$n_X^{(0)} = \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\frac{E_X}{kT}}$$

is the equilibrium density. Here, p denotes momentum of the particle, $E_X = \sqrt{p^2c^2 + m_X^2c^4}$ is the energy, and T is the common equilibrium temperature. We will only consider cases for which $mc^2 \gg kT$, i.e., systems for which the temperature is much lower than the rest mass of the particles.

You can assume as known that the photon density equals the equilibrium density during thermodynamic equilibrium, $n_\gamma = n_\gamma^{(0)}$.

a) We assume $n_e = n_p$, what is the assumption behind this equality? Define the free electron fraction to be

$$X_e = \frac{n_e}{n_e + n_H}.$$

and show that the Saha equation may be written on the form

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

$n_e = n_p$ tells us that there are an equal number of electrons and protons. The assumption behind $n_e = n_p$ is therefore that the Universe is electrically neutral. Using this assumption we get

$$\frac{n_e n_p}{n_H} = \frac{n_e^2}{n_H} = \frac{n_e^2}{(n_e + n_H)^2} \frac{(n_e + n_H)^2}{n_e + n_H - n_e} = \frac{n_e^2}{(n_e + n_H)^2} \frac{(n_e + n_H)}{1 - \frac{n_e}{n_e + n_H}} = X_e^2 \cdot \frac{(n_e + n_H)}{1 - X_e}$$

Applying Eq. 2 then gives us the result directly. The numerator, $n_e + n_H$ above equals $n_p + n_H = n_b$ - the numberdensity of baryons.

b) What is the usual approximation for the relativistic energy $E(p)$ in the non-relativistic limit ($m_Xc^2 \gg pc$)? Explain why is it valid (i.e. a good approximation) to use this approximation in the whole integral for $n_X^{(0)}$ (which includes arbitrary large p)? Show that the background density of (massive) species X is given by

$$n_X^{(0)} = \left(\frac{kTm_X}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{m_Xc^2}{kT}}.$$

Hint: You may need to know that

$$\int_0^\infty \sqrt{u}e^{-u}du = \int_0^\infty 2u^2 e^{-u^2}du = \frac{\sqrt{\pi}}{2}.$$

together with $d^3p = 4\pi p^2 dp$.

In this limit a Taylor expansion gives us $E = m_X c^2 + p^2/(2m_X)$ - the well-known rest energy + kinetic energy relation. Using this (valid since the integrand is practically zero when p is large enough to make this approximation wrong)

$$n_X^{(0)} = \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\frac{E_X}{kT}} = \int \frac{4\pi p^2 dp}{(2\pi\hbar)^3} e^{-\frac{m_X c^2 + p^2/(2m_X)}{kT}}$$

Making the substituion $u = p^2/(2m_X)$, using the provided integral and a bit of algebra gives the result.

c) Finally, show that the full Saha equation for the electron density is given by

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H}\left(\frac{kTm_e}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{\epsilon_0}{kT}}.$$

What is ϵ_0 here? Which assumption regarding m_p and m_H is used here? When do we naively expect recombination to happen? In our Universe recombination happens much later than our simple guess, what is the reason for this?

Evaluating $n_e n_p/n_H$ using the result from a) and b) together with $m_p/m_H \approx 1$ (this is true since the mass of hydrogen is largely just the mass of the proton) we get the desired result with $\epsilon_0 = m_p + m_e - m_H$. This is the ionization energy of hydrogen. We expect recombination when $\epsilon_0 \approx k_b T$ - when the mean temperature of photons drops below the ionization energy of hydrogen. It happens much later due to there being much more photons than baryons in our Universe (one to one billion) which means there will still be a lot of photons with higher energy than the mean that can ionize hydrogren. This manifest itself in the prefactor of the exponential being very large.

d) Why can't we use the Saha equation at all times, but must instead use the Peebles equation at late times? With the Peebles equation we don't have $X_e \to 0$ at late times as with Saha, but it converges to finite value. Explain this physically.

The Saha equation is only valid when the interaction is very effective. However at some point the interaction decouples and this is where we need the more accurate Peebles equation.

When the electron density gets very low (after the interaction decouples) and the Universe keeps expanding it will get harder and harder for the remaining electrons to find a proton to form hydrogen with. Thus the interaction eventually just stops and leaving us with (a freeze out abundance) a finite non-zero fraction.

Problem 3 – Initial conditions (20 points)

In order to solve the Boltzmann-Einstein equations, one needs initial conditions. When we set the initial conditions for our system we start early in the radiation era when the modes of interest have a comoving wavelength much larger than the comoving size of the horizon. In this problem you will derive the appropriate expressions for Ψ, δ, δ_b, Θ_0 and v, relating all to Φ. You should neglect neutrinos and polarisation in this problem, and you need only consider adiabatic initial conditions.

a) What does the ratio $k/\mathcal{H} \approx k\eta$ describe and what can we say about the size of this quantity when we set the initial conditions? How can we approximate $\mathcal{H}^2$ in the radiation era? What can we say about the size of the photon quadrupole Θ_2 in the early radiation era?

The ratio $k/\mathcal{H} = 2\pi r_H/\lambda$ is the ratio between the size of the horizon $r_H = 1/\mathcal{H}$ and the wave-length of the mode λ. If the wavelength is much larger than the size of the horizon then this ratio is much smaller than unity. When radiation is the dominating species we have $\mathcal{H}^2 \approx a^2 H_0^2(\Omega_{R0}/a^4) = H_0^2\Omega_{R0}/a^2$. In the early Universe photons are tightly coupled to baryons and higher order multipoles like the quadrupole are washed away so we can approximate $\Theta_2 \approx 0$.

b) Starting from the full set of perturbation equations listed in Appendix 1.3, write down a simplified (as much as possible) set of equations for $\Theta_0, \delta, \delta_b, \Phi$ and Ψ that is valid for very early times.

$$\Theta_0' = -\Phi'$$
$$\delta' = -3\Phi'$$
$$\delta_b' = -3\Phi'$$
$$\Phi' = \Psi + 2\Theta_0$$
$$\Psi = -\Phi$$

We ignored terms proportional to $k/\mathcal{H}$ and Θ_2 as per a). We used the approxumation for $\mathcal{H}$ in a). We ignored δ, δ_b in the Poisson equation as their prefactor is tiny prior to matter-radiation equality: $a^2\Omega_m/\Omega_r = (a/a_{\rm eq})^2 \ll 1$ (and they don't grow much before the matter era).

c) Derive a second-order differential equation for Φ as a function of a. Show that this equation has two solutions. Which solution survives and is therefore cosmologically relevant?

The derivative of the Poisson equation tells us

$$\Phi'' = -\Phi' + 2\Theta_0' = -3\Phi'$$

where we used $\Phi = -\Psi$ and the Θ_0 equation. This has solutions $\Phi =$ constant and $\Phi \propto e^{-3x} = 1/a^3$ which can for example by found by taking $\Phi = a^n = e^{nx}$, inserting into the equation and solving for n. The latter solution blows up as $a \to 0$ so it is not physical.

d) What do we mean by adiabatic initial conditions and what does this assumption imply for the relation between the different density contrasts? Find the appropriate initial conditions for Θ_0, δ, δ_b and Ψ expressed relative to Φ.

Adiabatic IC - the energy density of all matter matter components is the same as in the background at some slightly different time that varies from place to place: $(n_i/n_j)(t,x) = (\overline{n}_i/\overline{n}_j)(t + \delta t(x))$. This defines a relation between the different density contrasts $\delta_i/(1 + w_i) = \delta_j/(1 + w_j)$ which gives us $4\Theta_0/(4/3) = \delta = \delta_b$. Since $\Phi' = 0$ for the relevant solution in d) the Poisson equation also tells us that $2\Theta_0 = \Phi$ so $\delta = \delta_b = 3\Phi/2$. The final one is easy $\Psi = -\Phi$.

e) Use the equation for v' in Appendix 1.3 to derive the initial conditions for v.

Since $v \to 0$ as $a \to 0$ we cannot ignore the $k/\mathcal{H}$ term in the v equation as we did for the density perturbations. From $v' = -v + k/\mathcal{H} \cdot \Psi$ together with $\Psi = -\Phi =$ constant and $1/\mathcal{H} \propto a$ we get the ODE $v' + v \propto a$. A simple way to solve this is to make the guess $v = C(k/\mathcal{H})$ which inserted gives us $C = -C - \Phi$ and therefore $v = (k/\mathcal{H})/2 \cdot \Phi$.

Problem 4 – Line-of-sight integration (20 points)

In this problem, we will derive the expression for the transfer function, $\Theta_\ell(k)$, used in for line-of-sight integration. Before we begin, let us review some relations concerning the Legendre polynomials, $P_\ell(\mu)$, that you may or may not find useful in the following:

$$P_0(\mu) = 1$$
$$P_1(\mu) = \mu$$
$$P_\ell(\mu) = (-1)^\ell P_\ell(-\mu)$$
$$\int_{-1}^{1} P_\ell(\mu) P_{\ell'}(\mu) d\mu = \delta_{\ell\ell'} \frac{2}{2l+1}$$
$$j_\ell(x) = \frac{i^\ell}{2} \int_{-1}^{1} e^{-i\mu x} P_\ell(\mu) d\mu$$
$$f_\ell = \frac{i^\ell}{2} \int_{-1}^{1} f(\mu) P_\ell(\mu) d\mu$$

Here $j_\ell(x)$ is the spherical Bessel function of order ℓ, $f(\mu)$ is an arbitrary function of μ defined on $[-1, 1]$ and f_ℓ is its Legendre multipoles. Also, note that in the following, a dot $\dot{}$ means derivative with respect to conformal time.

a) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta],$$

where $\Theta = \Theta(k, \mu, \eta)$ and μ is the angle between the photon propagation direction, $\hat{p}$, and the wave vector, $\hat{k}$. For simplicity (and so that you don't have to calculate so much) we have ignored the term μv_b (i.e. we ignore the so-called Doppler contribution). Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\Theta_0,$$

and show that this equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S} e^{ik\mu(\eta - \eta_0) - \tau} d\eta.$$

Multiplying the equation by the integrating factor $e^{ik\mu(\eta-\eta_0)-\tau}$ we can write it as

$$\frac{d}{d\eta}[\Theta e^{ik\mu(\eta_0-\eta)-\tau}] = [-\dot{\Phi} - ik\mu\Psi - \dot{\tau}\Theta_0] e^{ik\mu(\eta-\eta_0)-\tau}$$

Integrating over $0 \to \eta_0$ using that $\tau \to \infty$ as $\eta \to 0$ and $\tau = 0$ when $\eta = \eta_0$ we get

$$\Theta(\eta_0, k) = \int [-\dot{\Phi} - ik\mu\Psi - \dot{\tau}\Theta_0] e^{ik\mu(\eta-\eta_0)-\tau} d\eta$$

where the first factor is S.

b) Assume that $\tilde{S}$ does not depend on μ (in this sub-problem only). Show that in this case

$$\Theta_\ell(\eta_0, k) = \int_0^{\eta_0} \tilde{S} e^{-\tau} j_\ell[k(\eta_0 - \eta)] d\eta,$$

where $\Theta_\ell(\eta, k)$ are the multipole expansion coefficients of $\Theta(\eta, k, \mu)$.

Multiplying

$$\Theta(\eta_0, k) = \int S e^{ik\mu(\eta-\eta_0)-\tau} d\eta$$

by $i^\ell/2 \cdot P_\ell(\mu)$ and integrating over $[-1, 1]$ using the definition of the multipoles and the spherical bessel-function provided in the problem assuming S is independent of μ we get

$$\Theta_\ell(\eta_0, k) = \int S e^{-\tau} j_\ell(k(\eta_0 - \eta)) d\eta$$

c) In reality, $\tilde{S}$ does of course depend on μ, and this have to be taken into account in the expression in b).

One way of doing this is to use the identity

$$ik\mu e^{ik\mu(\eta-\eta_0)} = \frac{d}{d\eta}[e^{ik\mu(\eta-\eta_0)}]$$

and then apply integration by parts to remove a factor of μ. This allows us to replace $ik\mu \leftrightarrow \frac{d}{d\eta}$ in the same way as with Fourier-transforms where we can replace $\frac{d}{dx} \leftrightarrow ik$ inside a transform.

Use this to show that the full solution for the transfer function is

$$\Theta_\ell(\eta_0, k) = \int_0^{\eta_0} S(k, \eta) j_\ell[k(\eta_0 - \eta)] d\eta,$$

where

$$S(k, \eta) = -\dot{\tau} e^{-\tau} [\Theta_0 + \Psi] + e^{-\tau}(\dot{\Psi} - \dot{\Phi})$$

First we write the equation as

$$\Theta(\eta_0, k) = \int [-\dot{\Phi} e^{-\tau} - \Psi e^{-\tau} \frac{d}{d\eta} - \dot{\tau} e^{-\tau} \Theta_0] e^{ik\mu(\eta-\eta_0)} d\eta$$

We use integration by parts on the middle term (boundary terms vanish or does not depend on μ so would only contribute to the monopole in the end) to get

$$\Theta(\eta_0, k) = \int [-\dot{\Phi} e^{-\tau} + \frac{d}{d\eta}(\Psi e^{-\tau}) - \dot{\tau} e^{-\tau} \Theta_0] e^{ik\mu(\eta-\eta_0)} d\eta$$

which becomes

$$\Theta(\eta_0, k) = \int [-\dot{\tau} e^{-\tau}[\Theta_0 + \Psi] + (\dot{\Psi} - \dot{\Phi}) e^{-\tau}] e^{ik\mu(\eta-\eta_0)} d\eta$$

where the terms in the brackets is seen to be $S(k,\eta)$. Note that this integral is now on the same form as in the previous problem (this "S" does not depend on μ) so the previous result gives us

$$\Theta_\ell(\eta_0,k) = \int [-\dot{\tau}e^{-\tau}[\Theta_0+\Psi] + (\dot{\Psi}-\dot{\Phi})e^{-\tau}]j_\ell(k(\eta_0-\eta))d\eta$$

Alternatively one can do the same steps as in the previous problem here to get the same result.

d) Sketch how the visibility function $g(\eta) \equiv -\dot{\tau}e^{-\tau}$ looks like and write down a simple analytical approximation for this function. Use this to perform the integral when the source function only consists of the first term

$$S(k,\eta) = g(\eta)[\Theta_0+\Psi](\eta,k)$$

Explain the result you get physically: what does this tell us about the CMB multipoles $\Theta_\ell(\eta_0,k)$ that we observe today? Describe what physical effect lies behind the term $e^{-\tau}(\dot{\Psi}-\dot{\Phi})$ that we ignored here.

g integrates to unity and is a sharp peak around recombination so we can approximate $g \approx \delta(\eta-\eta_{\rm rec})$. This gives us

$$\Theta_\ell(\eta_0,k) = [\Theta_0+\Psi](\eta_{\rm rec},k)j_\ell(k(\eta_0-\eta_{\rm rec}))$$

This tells us that the CMB anisotropies we observe today are the (effective) temperature inhomogenities that were present at the last scattering surface and which have free streamed to us today (and projected on a sphere; last factor). The term Ψ is there because photons had to climb the gravitational potentials they were created in to escape causing gravitational redshifting. The ISW term describes the effect of photons falling into and climbing out of devaying potentials (due to dark energy taking over). This changes the energy of the photons.

Problem 5 – Cosmological Parameters (20 points)

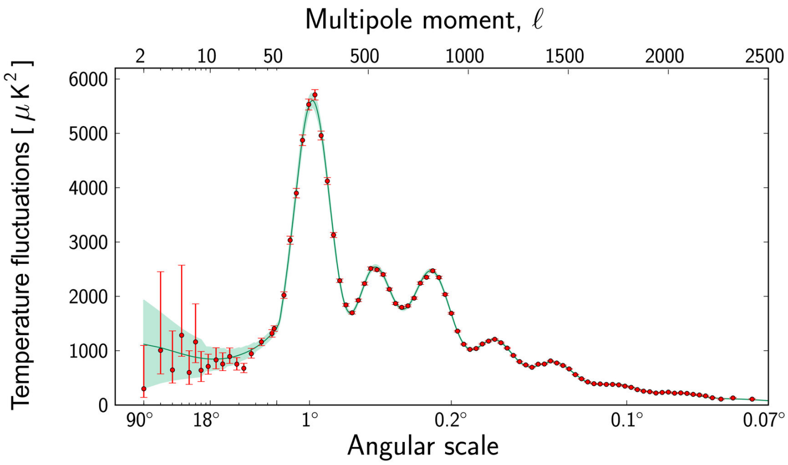

Figure 1: The best-fit ΛCDM Planck temperature power spectrum.

The figure above shows the CMB power spectrum. In the following we will consider how changes in the cosmological parameters affect the CMB spectrum. You just need to explain the main effect associated to the specific cosmological parameter assuming the other ones are kept constant.

a) What would happen to the CMB power spectrum if one increases Ω_b? Draw a cartoon to illustrate this. Explain this effect physically.

The main effect here is baryon loading (but also the diffusion damping will be modified). A high value of Ω_b means more baryonic matter. This leads to accoustic compressions being stronger than rarefications, since the extra baryons now starts to contribute themselves to the potential during compressions. This effect (baryon loading) implies that the first and third peak in the spectrum increases relative to the second and fourth peak. With a higher value of Ω_b the photons will collide more often reducing their mean-free-path and will therefore diffuse intro smaller regions that before. Thus the damping of the CMB will kick in at larger ℓ.

b) What would happen to the CMB power spectrum if one increases Ω_k? Draw a cartoon to illustrate this. Explain this effect physically.

If we change the curvature we mainly change the geometry of the late Universe. In a closed Universe the hot spots of the CMB will look larger than their actual size and in an open Universe they will look smaller. The CMB spectrum therefore shifts to the right. Mathematically: the angular size of the CMB is given by $\theta \propto r_s/D_A$ with D_A being the angular diameter distance. Increasing $\Omega_k > 0$ leads to a larger D_A which shifts the peaks to larger values of ℓ.

c) What would happen to the CMB spectrum if you *increase* the CMB temperature, T_0? Explain your reasoning.

If we increase T_0 then photons will generally by hotter. This means recombination will be delayed and happen closer to our present time. This makes the CMB hot spots appear larger so we would expect the spectrum to shift to the left.

d) What would happen to the CMB power spectrum if one increases A_s (primordial power spectrum amplitude) *and* the spectral tilt n_s? Draw a cartoon to illustrate this. What physical process determines the value of A_s and n_s?

If we change A_s we increase the initial amplitude of all modes. We have $C_\ell \propto A_s$ so this change translates into the power-spectrum shifting up (by a constant factor) for all ℓ. If we increase n_s then since $C_\ell \propto \ell^{n_s}$ it will tilt down for low ℓ and up for high ℓ. The initial fluctuations are generated in the very early Universe by inflation in our current paradigm. What time inflation happens (also called the energy scale of inflation - the temperature of the Universe when it happened) will (very roughly) determine the value A_s and how long inflation lasted determines n_s. The longer inflation lasted the closer n_s will to unity.

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{1}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{2}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{3}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.
- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{4}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\sum \rho_i, \quad \frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\sum(\rho_i + 3p_i), \tag{5}$$

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{6}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{7}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{8}$$

1.3 The perturbation equations

Einstein-Boltzmann equations (derivatives here are with respect to $x = \log(a)$):

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{9}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{10}$$

$$\Theta_\ell' = \frac{\ell k}{(2\ell+1)\mathcal{H}}\Theta_{\ell-1} - \frac{(\ell+1)k}{(2\ell+1)\mathcal{H}}\Theta_{\ell+1} + \tau'\left[\Theta_\ell - \frac{1}{10}\Theta_\ell\delta_{\ell,2}\right], \qquad \ell \geq 2 \tag{11}$$

$$\Theta_\ell' = \frac{k}{\mathcal{H}}\Theta_{\ell-1} - \frac{\ell+1}{\mathcal{H}\eta(x)}\Theta_\ell + \tau'\Theta_\ell, \qquad \ell = \ell_{\max} \tag{12}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{13}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{14}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{15}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{16}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{17}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{18}$$

1.4 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_\eta^{\eta_0} n_e\sigma_T a\, d\eta' \tag{19}$$

$$\tau' = -\frac{n_e\sigma_T a}{\mathcal{H}} \tag{20}$$

- Visibility function:

$$g(\eta) = -\dot{\tau}e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{21}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{22}$$

$$\int_0^{\eta_0} g(\eta)d\eta = \int_{-\infty}^0 \tilde{g}(x)dx = 1. \tag{23}$$

- The Saha equation,

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{24}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\text{K}/a$, and $\epsilon_0 = 13.605698\text{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b}\left[\beta(T_b)(1-X_e) - n_H\alpha^{(2)}(T_b)X_e^2\right], \tag{25}$$

  where

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{26}$$

$$\Lambda_{2s\to 1s} = 8.227\text{s}^{-1} \tag{27}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{28}$$

$$n_{1s} = (1-X_e)n_H \tag{29}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b} \tag{30}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{31}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{32}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b) \tag{33}$$

1.5 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' - \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{34}$$

2. The transfer function:

$$\Theta_\ell(k, x=0) = \int_{-\infty}^{0}\tilde{S}(k,x)j_\ell[k(\eta_0 - \eta(x))]dx \tag{35}$$

3. The CMB spectrum:

$$C_\ell = 4\pi\int_0^\infty A_s\left(\frac{k}{k_{\text{pivot}}}\right)^{n_s-1}\Theta_\ell^2(k)\frac{dk}{k} \tag{36}$$

This is the last page of the exam set.