UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220 — Cosmology II

Date: Wednesday, June 12th, 2024

Time: 15:00 – 19:00

The exam set consists of 11 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Each problem counts for 20% of the final score. Note that the problems 2-5 are ordered according to the order the topic was covered in the lectures and not by difficulty.

Problem 1 – Background questions (20 points)

Answer each question with at most three or four sentences.

a) Explain qualitatively how inflation gave rise to the fluctuations that seeded the subsequent growth of structure in the universe?

b) What is the equation of state for a perfect fluid? Mathematically what does it mean that the Universe is accelerating and what condition must the equation of state of a fluid responsible for an accelerated expansion of the Universe satisfy?

c) What are the main advantages of line of sight integration over the traditional Boltzmann hierarchy approach?

d) What is the visibility function and what is the physical interpretation of this? Why it is practically zero at very early times?

e) What do the Fermi-Dirac and Bose-Einstein distributions describe? What does the Maxwell distribution describe?

f) Why do we solve the Einstein-Boltzmann equations in Fourier-space instead of real-space?

g) What is the relationship between conformal time η and cosmic time t? Give a physical interpretation of conformal time.

h) The Einstein equation reads

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

What does the left- and right-hand sides of this equation describe, respectively? Why is it a good approximation to linearize this equation around a FRW metric when studying large scale structure formation?

Problem 2 – The Saha Equation (20 points)

The Saha equation plays a central role when calculating the CMB power spectrum. In the following, we will derive one form of this equation suitable for this purpose. For a gas consisting of photons, protons and electrons in thermodynamic equilibrium, one can show that the following relation, the Saha equation, holds

$$\frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} = \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}},$$

where n_X is the density of species X, and

$$n_X^{(0)} = \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\frac{E_X}{kT}}$$

is the equilibrium density. Here, p denotes momentum of the particle, $E_X = \sqrt{p^2c^2 + m_X^2c^4}$ is the energy, and T is the common equilibrium temperature. We will only consider cases for which $mc^2 \gg kT$, i.e., systems for which the temperature is much lower than the rest mass of the particles.

You can assume as known that the photon density equals the equilibrium density during thermodynamic equilibrium, $n_\gamma = n_\gamma^{(0)}$.

a) We assume $n_e = n_p$, what is the assumption behind this equality? Define the free electron fraction to be

$$X_e = \frac{n_e}{n_e + n_H}.$$

and show that the Saha equation may be written on the form

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

b) What is the usual approximation for the relativistic energy $E(p)$ in the non-relativistic limit ($m_Xc^2 \gg pc$)? Explain why is it valid (i.e. a good approximation) to use this approximation in the whole integral for $n_X^{(0)}$ (which includes arbitrary large p)? Show that the background density of (massive) species X is given by

$$n_X^{(0)} = \left(\frac{kTm_X}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{m_Xc^2}{kT}}.$$

Hint: You may need to know that

$$\int_0^\infty \sqrt{u}e^{-u}du = \int_0^\infty 2u^2e^{-u^2}du = \frac{\sqrt{\pi}}{2}.$$

together with $d^3p = 4\pi p^2dp$.

c) Finally, show that the full Saha equation for the electron density is given by

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \left(\frac{kTm_e}{2\pi\hbar^2} \right)^{3/2} e^{-\frac{\epsilon_0}{kT}}.$$

What is ϵ_0 here? Which assumption regarding m_p and m_H is used here? When do we naively expect recombination to happen? In our Universe recombination happens much later than our simple guess, what is the reason for this?

d) Why can't we use the Saha equation at all times, but must instead use the Peebles equation at late times? With the Peebles equation we don't have $X_e \to 0$ at late times as with Saha, but it converges to finite value. Explain this physically.

Problem 3 – Initial conditions (20 points)

In order to solve the Boltzmann-Einstein equations, one needs initial conditions. When we set the initial conditions for our system we start early in the radiation era when the modes of interest have a comoving wavelength much larger than the comoving size of the horizon. In this problem you will derive the appropriate expressions for Ψ, δ, δ_b, Θ_0 and v, relating all to Φ. You should neglect neutrinos and polarisation in this problem, and you need only consider adiabatic initial conditions.

a) What does the ratio $k/\mathcal{H} \approx k\eta$ describe and what can we say about the size of this quantity when we set the initial conditions? How can we approximate $\mathcal{H}^2$ in the radiation era? What can we say about the size of the photon quadrupole Θ_2 in the early radiation era?

b) Starting from the full set of perturbation equations listed in Appendix 1.3, write down a simplified (as much as possible) set of equations for $\Theta_0, \delta, \delta_b, \Phi$ and Ψ that is valid for very early times.

c) Derive a second-order differential equation for Φ as a function of a. Show that this equation has two solutions. Which solution survives and is therefore cosmologically relevant?

d) What do we mean by adiabatic initial conditions and what does this assumption imply for the relation between the different density contrasts? Find the appropriate initial conditions for Θ_0, δ, δ_b and Ψ expressed relative to Φ.

e) Use the equation for v' in Appendix 1.3 to derive the initial conditions for v.

Problem 4 – Line-of-sight integration (20 points)

In this problem, we will derive the expression for the transfer function, $\Theta_\ell(k)$, used in for line-of-sight integration. Before we begin, let us review some relations concerning the Legendre polynomials, $P_\ell(\mu)$, that you may or may not find useful in the following:

$$\begin{aligned} P_0(\mu) &= 1 \\ P_1(\mu) &= \mu \\ P_\ell(\mu) &= (-1)^\ell P_\ell(-\mu) \\ \int_{-1}^{1} P_\ell(\mu)P_{\ell'}(\mu)d\mu &= \delta_{\ell\ell'}\frac{2}{2l+1} \\ j_\ell(x) &= \frac{i^\ell}{2}\int_{-1}^{1} e^{-i\mu x}P_\ell(\mu)d\mu \\ f_\ell &= \frac{i^\ell}{2}\int_{-1}^{1} f(\mu)P_\ell(\mu)d\mu \end{aligned}$$

Here $j_\ell(x)$ is the spherical Bessel function of order ℓ, $f(\mu)$ is an arbitrary function of μ defined on $[-1,1]$ and f_ℓ is its Legendre multipoles. Also, note that in the following, a dot $\dot{}$ means derivative with respect to conformal time.

a) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta],$$

where $\Theta = \Theta(k,\mu,\eta)$ and μ is the angle between the photon propagation direction, $\hat{p}$, and the wave vector, $\hat{k}$. For simplicity (and so that you don't have to calculate so much) we have ignored the term μv_b (i.e. we ignore the so-called Doppler contribution). Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\Theta_0,$$

and show that this equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S} e^{ik\mu(\eta-\eta_0)-\tau} d\eta.$$

b) Assume that $\tilde{S}$ does not depend on μ (in this sub-problem only). Show that in this case

$$\Theta_\ell(\eta_0, k) = \int_0^{\eta_0} \tilde{S} e^{-\tau} j_\ell[k(\eta_0 - \eta)] d\eta,$$

where $\Theta_\ell(\eta, k)$ are the multipole expansion coefficients of $\Theta(\eta, k, \mu)$.

c) In reality, $\tilde{S}$ does of course depend on μ, and this have to be taken into account in the expression in b).

One way of doing this is to use the identity

$$ik\mu e^{ik\mu(\eta-\eta_0)} = \frac{d}{d\eta}[e^{ik\mu(\eta-\eta_0)}]$$

and then apply integration by parts to remove a factor of μ. This allows us to replace $ik\mu \leftrightarrow \frac{d}{d\eta}$ in the same way as with Fourier-transforms where we can replace $\frac{d}{dx} \leftrightarrow ik$ inside a transform.

Use this to show that the full solution for the transfer function is

$$\Theta_\ell(\eta_0, k) = \int_0^{\eta_0} S(k,\eta) j_\ell[k(\eta_0 - \eta)] d\eta,$$

where

$$S(k,\eta) = -\dot{\tau}e^{-\tau}\left[\Theta_0 + \Psi\right] + e^{-\tau}(\dot{\Psi} - \dot{\Phi})$$

d) Sketch how the visibility function $g(\eta) \equiv -\dot{\tau}e^{-\tau}$ looks like and write down a simple analytical approximation for this function. Use this to perform the integral when the source function only consists of the first term

$$S(k,\eta) = g(\eta)[\Theta_0 + \Psi](\eta, k)$$

Explain the result you get physically: what does this tell us about the CMB multipoles $\Theta_\ell(\eta_0, k)$ that we observe today? Describe what physical effect lies behind the term $e^{-\tau}(\dot{\Psi} - \dot{\Phi})$ that we ignored here.

Problem 5 – Cosmological Parameters (20 points)

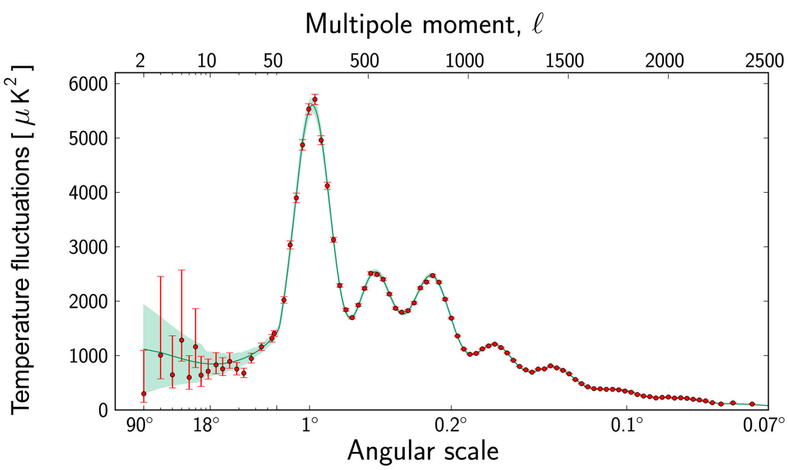

Figure 1: The best-fit ΛCDM Planck temperature power spectrum.

The figure above shows the CMB power spectrum. In the following we will consider how changes in the cosmological parameters affect the CMB spectrum. You just need to explain the main effect associated to the specific cosmological parameter assuming the other ones are kept constant.

a) What would happen to the CMB power spectrum if one increases Ω_b? Draw a cartoon to illustrate this. Explain this effect physically.

b) What would happen to the CMB power spectrum if one increases Ω_k? Draw a cartoon to illustrate this. Explain this effect physically.

c) What would happen to the CMB spectrum if you *increase* the CMB temperature, T_0? Explain your reasoning.

d) What would happen to the CMB power spectrum if one increases A_s (primordial power spectrum amplitude) *and* the spectral tilt n_s? Draw a cartoon to illustrate this. What physical process determines the value of A_s and n_s?

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{1}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{2}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{3}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{4}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\sum \rho_i, \quad \frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\sum(\rho_i + 3p_i), \tag{5}$$

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{6}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{7}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{8}$$

1.3 The perturbation equations

Einstein-Boltzmann equations (derivatives here are with respect to $x = \log(a)$):

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{9}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{10}$$

$$\Theta_\ell' = \frac{\ell k}{(2\ell+1)\mathcal{H}}\Theta_{\ell-1} - \frac{(\ell+1)k}{(2\ell+1)\mathcal{H}}\Theta_{\ell+1} + \tau'\left[\Theta_\ell - \frac{1}{10}\Theta_\ell\delta_{\ell,2}\right], \qquad \ell \geq 2 \tag{11}$$

$$\Theta_\ell' = \frac{k}{\mathcal{H}}\Theta_{\ell-1} - \frac{\ell+1}{\mathcal{H}\eta(x)}\Theta_\ell + \tau'\Theta_\ell, \qquad \ell = \ell_{\max} \tag{12}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{13}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{14}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{15}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{16}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{17}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{18}$$

1.4 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_\eta^{\eta_0} n_e\sigma_T a\, d\eta' \tag{19}$$

$$\tau' = -\frac{n_e\sigma_T a}{\mathcal{H}} \tag{20}$$

- Visibility function:

$$g(\eta) = -\dot{\tau}e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{21}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{22}$$

$$\int_0^{\eta_0} g(\eta)d\eta = \int_{-\infty}^0 \tilde{g}(x)dx = 1. \tag{23}$$

- The Saha equation,

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{24}$$

where $n_b = \frac{\Omega_b\rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\text{K}/a$, and $\epsilon_0 = 13.605698\text{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b}\left[\beta(T_b)(1-X_e) - n_H\alpha^{(2)}(T_b)X_e^2\right], \tag{25}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to1s}+\Lambda_\alpha}{\Lambda_{2s\to1s}+\Lambda_\alpha+\beta^{(2)}(T_b)}, \tag{26}$$

$$\Lambda_{2s\to1s} = 8.227\text{s}^{-1} \tag{27}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{28}$$

$$n_{1s} = (1-X_e)n_H \tag{29}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b} \tag{30}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{31}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{32}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b) \tag{33}$$

1.5 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0+\Psi+\frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi'-\Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{34}$$

2. The transfer function:

$$\Theta_\ell(k, x=0) = \int_{-\infty}^{0}\tilde{S}(k,x)j_\ell[k(\eta_0-\eta(x))]dx \tag{35}$$

3. The CMB spectrum:

$$C_\ell = 4\pi\int_0^\infty A_s\left(\frac{k}{k_{\text{pivot}}}\right)^{n_s-1}\Theta_\ell^2(k)\frac{dk}{k} \tag{36}$$

This is the last page of the exam set.