Home exam AST5220 / AST9420 Spring 2020

25 May 14:30 - 2 June 14:30

About the exam

There are six problems in this set. Five of the problems (1,2,3,4,5) are for AST5220 and five of the problems (2,3,4,5,6) are for AST9420. The maximum score you can get is 100 points.

Problem 1: Warm up (AST5220) [14 points]

This is a collection of some basic questions relating to things in this course. You should be brief, and answer with no more than 3-4 sentences per question.

a) [2 points] Why is the temperature of the cosmological neutrinos smaller than the temperature of the photons today?

Solution: The neutrinos decoupled before electron/position annihilation which deposited energy into the primordial plasma and increased the temperature of the photons.

b) [2 points] What is the equation of state for a perfect fluid? Mathematically what does it mean that the Universe is accelerating and what condition must the equation of state of a fluid responsible for an accelerated expansion of the Universe satisfy?

Solution: The equation of state is the ratio between pressure and energy density. Acceleration means $\ddot{a} > 0$. From the second Fridmann equation $\ddot{a}/a = -\frac{4\pi G}{3}(\rho + 3p)$ we see that we must have $1 + 3w < 0$ in order for $\ddot{a} > 0$ so $w < -1/3$.

c) [2 points] What are the main reason(s) we solve the CMB equations in Fourier space instead of doing it in real space?

Solution: The Fourier transform transforms differentials into algebraic expressions. This means we end up having to solve simple ODEs in time instead of more complicated PDEs in space and time. In linear perturbation theory the different modes evolve independently so we can solve the equation set mode for mode (and only solve for the modes we are interested in).

d) [2 points] A new physical effect is found that causes the theoretical CMB quadrupole (C_ℓ for $\ell = 2$) to increase by 1% while keeping everything else the same. Explain why would this almost be impossible to detect in a CMB experiment?

Solution: Cosmic variance. (We have only one Universe to do measurements in and the initial conditions are that of a stochastic field. For the largest modes we have very few samples to accurately determine the underlying power-spectrum. The variance associated with C_2 is much larger than 1% so such a small effect would not be possible to measure.)

e) [2 points] What are the main advantages of line of sight integration over the traditional Boltzmann hierarchy approach?

Solution: It reduces the amount of ℓ's we need to include when solving the Boltzmann hierarchy from $1000 - 2000$ down to ~ 10 or so. It also has the advantage of making it more easy to see how the evolution of the perturbations maps onto the angular power-spectrum.

f) [2 points] What is the visibility function and what is the physical interpretation of this? Why it is practically zero at very early times?

Solution: The visibility function is defined as $g = \frac{d}{d\eta}e^{-\tau}$. It satisfies $\int g = 1$. It represents the probability density for a photon last scattering at time η. It has a peak at the last scattering surface. It is zero at very early times because its very improbable that a photon made it from the very early Universe till today without scattering at all (photons scattered many times in the tight coupling regime). At late times the Universe gets reionized, this causes some of the photons free-streaming towards us from the LSS to scatter so we get another smaller (and broader) peak in the visibility function.

g) [2 points] In the Boltzmann equation for photons we encountered the term

$$\frac{\partial f}{\partial \hat{p}}\frac{d\hat{p}}{dt}$$

Explain why we can ignore this term when deriving an equation for the photon perturbation Θ. If this term could not be ignored then how would you compute $\frac{d\hat{p}}{dt}$? (Don't do the calculation)

Solution: To zeroth order the photon distribution function does not depend on direction only on the magnitude of the momenta. Thus the first factor is (at least) a first order term. At zeroth order (at the background level) the direction of a photon does not change so the second term is also (at least) a first order term. Combined this is therefore a second order term can be ignored to first order in perturbation theory. The term $\frac{d\hat{p}}{dt}$ follows from the geodesic equation.

Problem 2: The CMB and matter power-spectrum (AST5220/AST9420) [20 points]

This problem is about understanding features in the CMB and matter power-spectrum. You might need to do some very small calculations in a few questions, but its mainly about the physical understanding. Keep the answers brief.

a) [4 points] The line of sight integration expression for Θ_ℓ (ignoring quadrupolar corrections) is given by

$$\Theta_\ell(k,\eta_0) = \int_0^{\eta_0} \left[g(\Theta_0+\Psi) + \left(\frac{d\Psi}{d\eta} - \frac{d\Phi}{d\eta}\right)e^{-\tau} - \frac{1}{k}\frac{d}{d\eta}(gv_b)\right] j_\ell(k\eta_0 - k\eta)d\eta$$

Explain what the different terms above represent physically. Use what you know about the visibility function to write down an approximate expression for the integral of the first term. How do we explain this result physically, i.e. what does this tell us about how the anisotropies we observe today are formed?

Solution: The first is the SW term: photons have to climb out of the gravitational potential to reach us today and this changes the temperature of the photons. The second is the ISW term: change in energy of photons are they travel in time-evolving gravitational wells. This splits into an early and late part. The third is the Doppler term (in tight coupling $v_b \sim -3\Theta_1 = v_\gamma$ the photon 'velocity' so this term is closely related to the photon velocity). The visibility function peaks sharply around recombination and is close to zero otherwise (with the exception of reionization). A rough approximation is therefore $g \approx \delta(\eta - \eta_*)$. This gives the contribution

$$(\Theta_0+\Psi)(\eta_*,k)j_\ell(k\eta_0 - k\eta_*) \approx (\Theta_0+\Psi)(\eta_*,k)j_\ell(k\eta_0)$$

This expression tells us that the CMB anisotropies we observe today come from temperature inhomogenities at the last scattering surface. The perturbations that are created by inflation are processed in the early Universe and frozen in at the last scattering surface. From there the photons free-streams to us. The $j_\ell(k\eta_0)$ term represents the free-streaming together with a projection on a sphere.

b) [4 points] You are given an theoretical spectrum showing the Sachs-Wolfe plateau ($\ell < 30$) of the CMB power spectrum. C_ℓ is seen to increase as we go from large to small ℓ. What are the cosmological parameters and the related physical effects that can cause such a signal?

Solution: Inflation can cause this (imprints larger fluctuations on large scales in the initial conditions): a value of $n_s < 1$ would cause the power-spectrum to tilt up at low ℓ. Dark energy Ω_Λ via the late-time integrated Sachs-Wolfe effect (decay of gravitational potentials as the Universe starts to accelerate) can also cause this. We also have the same effect when the Universe is curvature dominated. A large value of the optical depth can also cause a similar effect (it washes out anisotropies on smaller scales making C_ℓ larger at low ℓ than at larger ℓ's). It will however not curve upwards, but flatten, as we go to low ℓ. Thus its a sign of $n_s < 1$, $\Omega_\Lambda > 0$, $\Omega_K < 0$ (or τ).

c) [4 points] Explain briefly the effects of changing the optical depth of reionizaiton on the CMB (including polarization) and matter power-spectrum. Give an example of a cosmological parameter the optical depth is degenerate with in the CMB power-spectrum.

Solution: The optical depth of reionization has very little effect on the matter power-spectrum (it has an indirect effect on very small scales, but that is not relevant for this course), it only affects photons and photons have negligible effect on density perturbations in the late Universe. The effect on the CMB power-spectrum is to wash out the anisotropies, i.e. earlier reionization leads to a larger optical depths and this reduces the amplitude of the CMB spectrum. It is strongly degenerate with the primordial amplitude A_s as this also modulates the amplitude of the CMB spectrum. It causes a "bump" at low ℓ's in the polarization power spectra.

d) [4 points] The evolution equation for tensor perturbation h are given by

$$\frac{d^2h}{d\eta^2} + 2\frac{1}{a}\frac{da}{d\eta}\frac{dh}{d\eta} + k^2h = 0$$

What kind of equation is this? The tensor perturbation h acts as a source for tensor perturbations in the photon distribution which adds to the scalar perturbation signal we computed in the lectures for the CMB power-spectrum.

It only sources these only for scales where the amplitude h is sizable in the period around recombination. On the basis of how h evolves explain why we would not see any effects of tensor perturbations in the temperature power-spectrum for large ℓ.

Solution: Its a damped wave equation (damped harmonic oscillator) so the solutions will be damped oscillations. As soon a mode enters the horizon it will start to decay and undergo damped oscillations around $h = 0$. The earlier a mode enters the horizon (small scales modes $\to$ large ℓ's) the more damped it will be at the time of recombination when the temperature perturbations are frozen in.

e) [4 points] The matter power-spectrum today

$$P(k) = \Delta_M^2(a = 1, k) P_{\text{priomordial}}(k)$$

where $P_{\text{priomordial}}(k) = \frac{2\pi^2}{k^3} A_s (k/k_{\text{pivot}})^{n_s - 1}$ is the primordial power-spectrum and $\Delta_M(a, k) \equiv \frac{\Phi(a,k)k^2}{4\pi G\bar{\rho}(a)a^2}$ is the matter density contrast has a peak around $k \sim 0.01h/\text{Mpc}$. Explain the reason for why we get this peak based on how Δ_M grows in different regimes and give an expression for the wave-number k_{peak} it correspond to? Use this to discuss what cosmological parameter combination the position of the peak depends on (you can assume that the only relevant forms of energy in the Universe is matter and radiation and you don't need to derive how Δ_M evolves in different regimes - its enough to explain how it evolves).

Solution: In the matter era modes outside and inside the horizon grows as $\propto a$. In the radiation era modes outside the horizon grows as $\propto a^2$, but inside the horizon it only grows as $\propto \log(a)$. A mode enter the horizon when $k = \mathcal{H}$. Thus modes that enter before matter-radiation will be suppressed roughly by a factor $(a_{\text{enter}}/a_{\text{eq}})^2$ compared to modes that enter after a_{eq}. Thus the power-spectrum on scales $k > k_{\text{eq}}$ is suppressed roughly by a factor $(k_{\text{eq}}/k)^4$. This critical scale is the equality scale $k_{\text{eq}} = \mathcal{H}(a_{\text{eq}}) = H_0\sqrt{\Omega_M/a_{\text{eq}} + \Omega_R/a_{\text{eq}}^2} = H_0\sqrt{2\Omega_M^2/\Omega_R}$ where we have used $a_{\text{eq}} = \frac{\Omega_R}{\Omega_M}$ to simplify. The position of the peak therefore depends on $\frac{(\Omega_M h^2)^2}{(\Omega_R h^2)}$ (but the denominator is fixed by the CMB temperature which we know very accurately so it mainly depends on the physical matter density $\Omega_M h^2$.)

Problem 3: Scattering processes involving baryons (AST5220/AST9420) [10 points]

When you derived the Boltzmann equation for baryons, you found one equation for the baryon density and one equation for the average baryon velocity. Give a short physical/intuitive explanation for what each of these two equations tell us.

Consider all these scattering processes involving baryons:

1. $p^+\gamma \to p^+\gamma$
2. $e^-\gamma \to e^-\gamma$
3. $e^-e^- \to e^-e^-$
4. $e^-e^+ \leftrightarrow \gamma\gamma$
5. $e^-p^+ \to e^-p^+$
6. $n \to e^-p^+\bar{\nu}_e$
7. $e^-\nu_e \to e^-\nu_e$

where p^+ denotes a proton, e^- an electron, e^+ a positron, n a neutron, ν_e an electron neutrino and γ a photon.

AST5220: For each of these scattering processes give a short physical/intuitive explanation for why that process would contribute or not contribute (or be negligible) to the collision term for the equation for baryon density.

AST9420: For each of these scattering processes give a short physical/intuitive explanation for why that process would contribute or not contribute (or be negligible) to the collision term for the equation for baryon density and/or the equation for baryon velocity.

Note that process number 4 can go in both directions, consider the process in each direction separately.

Solution: Let us call the equation for density A and the equation for the velocity B.

1. $p^+\gamma \to p^+\gamma$: Does not change particle number so does not contribute to A, but it will contribute to B. The process $e^-\gamma \to e^-\gamma$ dominates however since $\sigma_T \sim 1/m^2$.
2. $e^-\gamma \to e^-\gamma$: This is normal Thompson scattering. It does not change particle number so does not contribute to A, but it will contribute to B since e^- could change it's momentum in the collision.
3. $e^-e^- \to e^-e^-$: Does not change particle number so does not contribute to A. Momentum conservation ensures that the average momentum of the baryon fluid is not changed during the collision, which means that it does not contribute to B either.
4. $e^-e^+ \leftrightarrow \gamma\gamma$: This would create or destroy electrons and positrons which would contribute to A, and possibly also B. However, close to recombination all the positrons have already annihilated, so the process $e^-e^+ \to \gamma\gamma$ cannot happen, and the photons do not have enough energy to create an electron positron pair so $\gamma\gamma \to e^-e^+$ cannot happen either.
5. $e^-p^+ \to e^-p^+$: Does not change particle number so does not contribute to A. Momentum conservation ensures that the average momentum of the baryon fluid is not changed during the collision, which means that it does not contribute to B either. This interaction is the one that keeps the baryon fluid together though, so in a sense the whole equation for the baryon fluid implicitly depends on this process.
6. $n \to e^-p^+\bar{\nu}_e$: This would create electrons and protons which would contribute to both A and B. However, close to recombination all the neutrons have already decayed, so the process does not happen anymore.
7. $e^-\nu_e \to e^-\nu_e$: Does not change particle number so does not contribute to A. It would contribute to B in a way similar to what $e^-\gamma \to e^-\gamma$ would do, however, even though there are plenty of electrons and neutrinos, electrons and neutrinos only interact through the weak force, so the cross section for this process is negligible.

Problem 4: Recombination including Helium (AST5220/AST9420) [26 points]

In our treatment of recombination in the lectures we assumed there only was hydrogen in our Universe. However there is also a sizable amounts of helium in our Universe. Helium comes in the forms He^0, He^+ and He^{++} (i.e. neutral, singly ionized an doubly ionized helium) with corresponding masses m_{He^0}, m_{He^+} and $m_{\text{He}^{++}}$. The mass ratio of helium to hydrogen that is generated in big bang nucleosynthesis is given by

$$Y_p \equiv \frac{4n_{\text{He}}}{n_B}$$

where $n_{\text{He}} = n^0_{\text{He}} + n^+_{\text{He}} + n^{++}_{\text{He}}$. Likewise hydrogen comes in the two forms H^0 and H^+ and $n_{\text{H}} = n_{\text{H}^0} + n_{\text{H}^+}$ with corresponding masses m_{H^0} and m_{H^+} and the total baryon number density is given by $n_B = 4n_{\text{He}} + n_{\text{H}}$ (which basically is the number density of protons/neutrons). In our Universe $Y_p \approx \frac{1}{4}$.

The interactions between helium, hydrogen and electrons relevant for recombination is the following:

$$\text{He}^{++} + e^- \rightleftharpoons \text{He}^+ + \gamma$$

$$\text{He}^+ + e^- \rightleftharpoons \text{He}^0 + \gamma$$

$$\text{H}^+ + e^- \rightleftharpoons \text{H}^0 + \gamma$$

You are in this problem going to derive a closed system of equations for recombination including both hydrogen and helium in the Saha approximation. You can use without proof that the equilibrium distribution $n_i^{\text{Eq}} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-\frac{m_i - \mu_i}{T}}$ for a non-relativistic particle in the low temperature limit and that $g_e = 2$, $\frac{g_{\text{He}^{++}}}{g_{\text{He}^+}} = 2$, $\frac{g_{\text{He}^+}}{g_{\text{He}^0}} = 1$ and $\frac{g_{\text{H}^+}}{g_{\text{H}^0}} = \frac{1}{2}$.

a) [6 points] Write down the Saha equations for the first two interactions above. You are free to assume that the interactions are in chemical equilibrium such that $\mu_1 + \mu_2 = \mu_3 + \mu_4$ for a $1 + 2 \rightleftharpoons 3 + 4$ process. Simplify the equations you find where possible and explain the approximations you are making. The Saha equation for hydrogen, the last ineraction above, was derived in the lectures and the result we found was

$$\frac{n_{\text{H}^+} n_{\text{e}^-}}{n_{\text{H}^0}} = \frac{g_{\text{H}^+} g_e}{g_{\text{H}^0}} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

where ϵ_0 is the ionization energy of hydrogen.

Solution:

$$\frac{n_{\text{He}^{++}} n_{\text{e}^-}}{n_{\text{He}^+}} = \frac{g_{\text{He}^{++}} g_e}{g_{\text{He}^+}} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\chi_1}{T}} = 4 \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\chi_1}{T}}$$

$$\frac{n_{\text{He}^+} n_{\text{e}^-}}{n_{\text{He}^0}} = \frac{g_{\text{He}^+} g_e}{g_{\text{He}^0}} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\chi_0}{T}} = 2 \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\chi_0}{T}}$$

$$\frac{n_{\text{H}^+} n_{\text{e}^-}}{n_{\text{H}^0}} = \frac{g_{\text{H}^+} g_e}{g_{\text{H}^0}} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

We have here assumed that $\frac{m_{\text{He}^{++}}}{m_{\text{He}^+}} \approx \frac{m_{\text{He}^+}}{m_{\text{He}^0}} \approx 1$ in the prefactor (but not in the exponential!) and we have defined $\chi_1 = m_{\text{He}^{++}} + m_{\text{e}} - m_{\text{He}^+}$ the ionization energy for singly ionized helium, $\chi_0 = m_{\text{He}^+} + m_{\text{e}} - m_{\text{He}^0}$ the ionization energy for neutral helium and $\epsilon_0 = m_{\text{H}^+} + m_{\text{e}} - m_{\text{H}^0}$ the ionization energy for hydrogen. We have assumed that $n_\gamma = n_\gamma^{\text{Eq}}$, i.e. photons stay in equilibrium with the heat bath. We have used that $\mu_\gamma = 0$, i.e. photons have zero chemical potential.

b) [6 points] Introduce the ionization fractions $x_{\text{He}^{++}} = \frac{n_{\text{He}^{++}}}{n_{\text{He}}}$, $x_{\text{He}^+} = \frac{n_{\text{He}^+}}{n_{\text{He}}}$ and $x_{\text{H}^+} = \frac{n_{\text{H}^+}}{n_{\text{H}}}$ and show that the system above can be written as a closed system for the three x_i's (i.e. the equations should only depend on T, n_{e^-}, the x_i's and physical constants). For hydrogen the result we derived in the lectures reads

$$\frac{x_{\text{H}^+} n_{\text{e}^-}}{1 - x_{\text{H}^+}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

Solution: We can write this as

$$\frac{x_{\mathrm{He}^{++}} n_{\mathrm{e}^-}}{x_{\mathrm{He}^+}} = 4\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\chi_1}{T}}$$

$$\frac{x_{\mathrm{He}^+} n_{\mathrm{e}^-}}{\frac{n_{\mathrm{He}^0}}{n_{\mathrm{He}}}} = 2\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\chi_0}{T}}$$

$$\frac{x_{\mathrm{H}^+} n_{\mathrm{e}^-}}{\frac{n_{\mathrm{H}^0}}{n_{\mathrm{H}}}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

Using $n_{\mathrm{He}} = n_{\mathrm{He}}^0 + n_{\mathrm{He}}^+ + n_{\mathrm{He}}^{++}$ and $n_{\mathrm{H}} = n_{\mathrm{H}}^0 + n_{\mathrm{H}}^+$ in the last two equations to eliminate n_{He^0} and n_{H^0} we arrive at

$$\frac{x_{\mathrm{He}^{++}} n_{\mathrm{e}^-}}{x_{\mathrm{He}^+}} = 4\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\chi_1}{T}}$$

$$\frac{x_{\mathrm{He}^+} n_{\mathrm{e}^-}}{1 - x_{\mathrm{He}^+} - x_{\mathrm{He}^{++}}} = 2\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\chi_0}{T}}$$

$$\frac{x_{\mathrm{H}^+} n_{\mathrm{e}^-}}{1 - x_{\mathrm{H}^+}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

If we specify T and n_{e^-} the rest of the numbers are known and this therefore represents a closed system of equations for the x_i's.

c) [6 points] The final part is to obtain an expression for the electron number density. What is the number density of free electrons in terms of $n_{\mathrm{He}}^+, n_{\mathrm{He}}^{++}, n_{\mathrm{H}}^+$ and what is the assumption you are making to get this? Use this to express n_{e^-} in terms of n_B, i.e. $n_{\mathrm{e}^-} = F n_B$ for some F depending on the x_i's. Use this to express the system above in terms of n_B and F and write it on such a way that the left hand side depends only depends on the x_i's (and F) and the right hand side consists only of known quantities. As a check that you got the correct result show that when there is no helium in the Universe (i.e. $Y_p = 0$ and in this case $F = x_{\mathrm{H}^+}$) then one of the equations above reduces to the result we derived in the lectures

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_B}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

where $X_e = \frac{n_{\mathrm{e}^-}}{n_{\mathrm{H}}}$.

Solution: The Universe is (and remains) electrically neutral so charge conservation implies $n_{\mathrm{e}^-} = 2n_{\mathrm{He}}^{++} + n_{\mathrm{He}}^+ + n_{\mathrm{H}}^+$. We can write this as

$$n_{\mathrm{e}^-} = 2\frac{n_{\mathrm{He}^{++}}}{n_{\mathrm{He}}} n_{\mathrm{He}} + \frac{n_{\mathrm{He}^+}}{n_{\mathrm{He}}} n_{\mathrm{He}} + \frac{n_{\mathrm{H}^+}}{n_{\mathrm{H}}} n_{\mathrm{H}}$$

$$n_{\mathrm{e}^-} = 2x_{\mathrm{He}^{++}} n_{\mathrm{He}} + x_{\mathrm{He}^+} n_{\mathrm{He}} + x_{\mathrm{H}^+} n_{\mathrm{H}}$$

$$n_{\mathrm{e}^-} = 2x_{\mathrm{He}^{++}} \frac{n_{\mathrm{He}}}{n_{\mathrm{B}}} n_{\mathrm{B}} + x_{\mathrm{He}^+} \frac{n_{\mathrm{He}}}{n_{\mathrm{B}}} n_{\mathrm{B}} + x_{\mathrm{H}^+} \frac{n_{\mathrm{H}}}{n_{\mathrm{B}}} n_{\mathrm{B}}$$

$$n_{\mathrm{e}^-} = \left[(2x_{\mathrm{He}^{++}} + x_{\mathrm{He}^+})\frac{Y_p}{4} + x_{\mathrm{H}^+}(1 - Y_p)\right] n_{\mathrm{B}}$$

where we have used the definition $Y_p \equiv \frac{4n_{\mathrm{He}}}{n_B}$. Thus

$$F = \left[(2x_{\mathrm{He}^{++}} + x_{\mathrm{He}^+})\frac{Y_p}{4} + x_{\mathrm{H}^+}(1 - Y_p)\right]$$

This gives us

$$F\frac{x_{\mathrm{He}^{++}}}{x_{\mathrm{He}^+}} = 4\frac{1}{n_B}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\chi_1}{T}}$$

$$F\frac{x_{\mathrm{He}^+}}{1 - x_{\mathrm{He}^+} - x_{\mathrm{He}^{++}}} = 2\frac{1}{n_B}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\chi_0}{T}}$$

$$F\frac{x_{\mathrm{H}^+}}{1 - x_{\mathrm{H}^+}} = \frac{1}{n_B}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

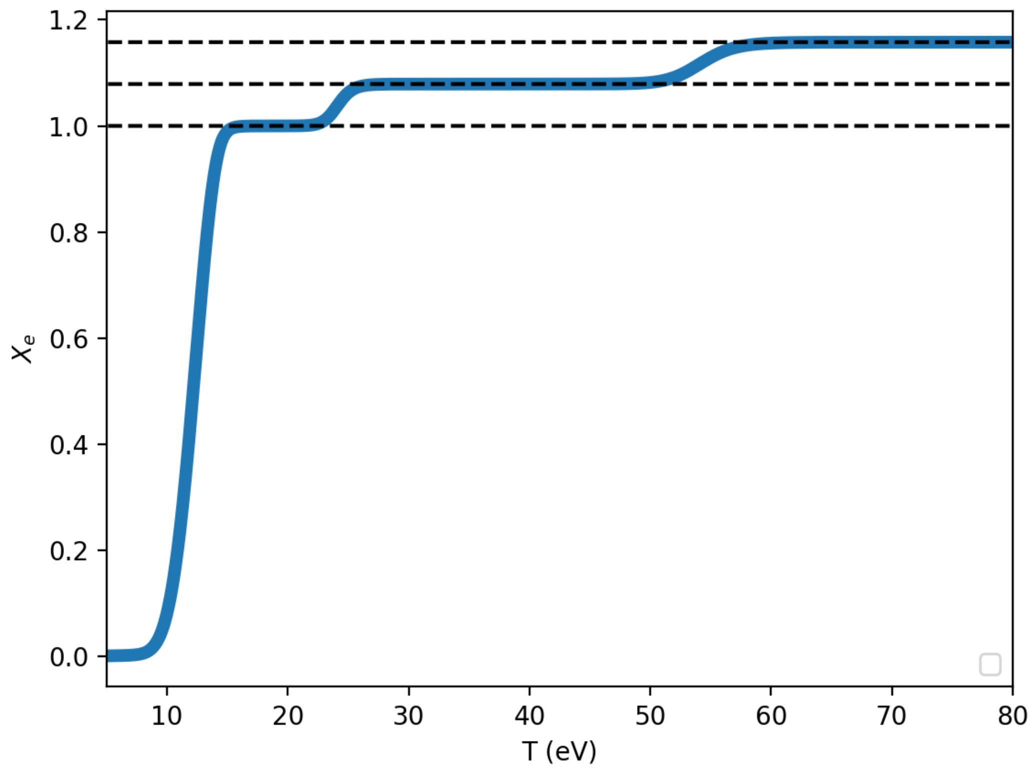

Figure 1: Evolution the free electron fraction as function of temperature. This is just a sketch and the numbers does not correspond to when the transitions happen in our Universe.

When there is no helium we have $F = x_{\rm H^+}$ and the last equation above gives us

$$\frac{x_{\rm H^+}^2}{1 - x_{\rm H^+}} = \frac{1}{n_B}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

Since $n_{\rm H^+} = n_{\rm e^-}$ we have that $x_{\rm H^+} = X_e$.

d) [8 points] What we are mainly interested in for recombination is the free electron fraction $X_e \equiv \frac{n_{\rm e^-}}{n_H}$ as this determines the optical depth. In Figure 1 we show a sketch for the evolution of X_e as function of temperature. Explain from the equations you have derived (and give a physical reason for it) this evolution; what do the three regimes seen in the figure correspond to? If you did not manage the last problem try to explain the evolution on purely physical grounds.

Derive a formula for the value of X_e in terms of Y_p at the three plateaus you see (the three dashed lines). For this discussion you can use that $m_{\rm He^{++}} + m_{\rm e} - m_{\rm He^+} \approx 55$ eV, $m_{\rm He^+} + m_{\rm e} - m_{\rm He^0} \approx 25$ eV and $\epsilon_0 = m_{\rm H^+} + m_{\rm e} - m_{\rm H^0} \approx 13$ eV. Hint: You are not meant to try to solve the equations exactly, but rather find approximate solutions for the x_i's in the different regimes either from the equations or from physical reasoning. Recall that if $T \ll T_*$ then $e^{-T_*/T} \approx 0$.

Solution: For temperatures above 55 eV all the exponentials are ≈ 1 and the right hand side of all the equations are large so $x_{\rm He^{++}} + x_{\rm He^+} \approx 1$ and $x_{\rm He^{++}}/x_{\rm He^+} \gg 1$ and $x_{\rm H^+} \approx 1$. Thus $x_{\rm He^{++}} \approx 1$, $x_{\rm He^+} \approx 0$ and $x_{\rm H^+} \approx 1$. Physically: the temperature is so large that everything is completely ionized!

As the temperature drops below 55 eV the exponential $e^{-\frac{\chi_1}{T}} \to 0$ so $x_{\rm He^{++}} \to 0$ and $x_{\rm He^+} \to 1$. Physically: photons do not have enough energy to ionize $\rm He^+$, but the temperature is still too large for neutral helium to form.

For temperatures below 25 eV $x_{\rm He^+} \to 0$ and still $x_{\rm H^+} \approx 1$. Physically: photons no longer have the ability to ionize helium so we end up with completely neutral helium, but hydrogen is still completely ionized.

For temperatures below 13 eV photons no longer have the ability to ionize hydrogen so $x_{\rm H^+} \to 0$.

In the different regimes above we find $X_e \approx \frac{1-Y_p/2}{1-Y_p}$ for $T > 55$ eV, $X_e \approx \frac{1-3Y_p/4}{1-Y_p}$ for $55 > T > 25$ eV, $X_e \approx 1$ for $25 > T > 13$ eV and $X_e \to 0$ for $T < 13$ eV. We are here cheating a bit and forgetting about the size of the photon-to-baryon number. If this is large then this will reduce the temperature for when these transitions actually happen (roughly by a factor of 40 in our Universe), but the general picture stays the same.

Problem 5: Vector perturbations (AST5220/AST9420) [30 points]

In the course you have studied both scalar and tensor perturbations, but now we will take a closer look at vector perturbations and how they evolve in an expanding universe.

A general 3D vector field, v_i, can always be decomposed into two parts, the part with no curl, and the part with no divergence:

$$v_i = v_i^{\parallel} + v_i^{\perp},$$

where $\nabla \times \mathbf{v}^{\parallel} = 0$ and $\nabla \cdot \mathbf{v}^{\perp} = 0$. The first part, $\mathbf{v}^{\parallel} = \nabla\phi$, can always be written as the gradient of some scalar field, ϕ and is therefore considered the scalar part of v_i[1]. The latter part of the velocity, $v_i^{\perp}$, is what is called the vector component of v_i and it is what we will consider in this problem.

We will only work with perturbations to first order in this problem, so you can safely neglect all higher order terms.

a) [4 points] Consider a general perturbed metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu}$ is a small perturbation and a $^{-}$ denotes an unperturbed quantity.

The Christoffel symbols are given by:

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left[\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\lambda\nu}\right].$$

Let us define the perturbed Christoffel symbols $\delta\Gamma^{\mu}_{\nu\lambda}$ by the following equation

$$\Gamma^{\mu}_{\nu\lambda} = \bar{\Gamma}^{\mu}_{\nu\lambda} + \delta\Gamma^{\mu}_{\nu\lambda}.$$

Show that to first order in the perturbation $h_{\mu\nu}$

$$\delta\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}\bar{g}^{\mu\rho}\left[\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\lambda\nu} - 2h_{\rho\sigma}\bar{\Gamma}^{\sigma}_{\nu\lambda}\right].$$

Hint: $h^{\mu\nu} \equiv g^{\mu\nu} - \bar{g}^{\mu\nu} = -\bar{g}^{\mu\rho}h_{\rho\sigma}\bar{g}^{\sigma\nu} \neq \bar{g}^{\mu\rho}h_{\rho\sigma}\bar{g}^{\sigma\nu}$.

Solution:

$$\begin{aligned}
\Gamma^{\mu}_{\nu\lambda} &= \frac{1}{2}\left(\bar{g}^{\mu\rho} + h^{\mu\rho}\right)\left[\partial_\lambda \bar{g}_{\rho\nu} + \partial_\nu \bar{g}_{\rho\lambda} - \partial_\rho \bar{g}_{\lambda\nu}\right] + \frac{1}{2}\Big(\bar{g}^{\mu\rho} + \underbrace{h^{\mu\rho}}_{\text{2. order}}\Big)\left[\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\lambda\nu}\right] \\
&= \underbrace{\frac{1}{2}\bar{g}^{\mu\rho}\left[\partial_\lambda \bar{g}_{\rho\nu} + \partial_\nu \bar{g}_{\rho\lambda} - \partial_\rho \bar{g}_{\lambda\nu}\right]}_{\bar{\Gamma}^{\mu}_{\nu\lambda}} + \frac{1}{2}h^{\mu\rho}\left[\partial_\lambda \bar{g}_{\rho\nu} + \partial_\nu \bar{g}_{\rho\lambda} - \partial_\rho \bar{g}_{\lambda\nu}\right] + \frac{1}{2}\bar{g}^{\mu\rho}\left[\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\lambda\nu}\right] \\
&= \bar{\Gamma}^{\mu}_{\nu\lambda} - \bar{g}^{\mu\alpha}h_{\alpha\sigma}\underbrace{\frac{1}{2}\bar{g}^{\sigma\rho}\left[\partial_\lambda \bar{g}_{\rho\nu} + \partial_\nu \bar{g}_{\rho\lambda} - \partial_\rho \bar{g}_{\lambda\nu}\right]}_{\bar{\Gamma}^{\sigma}_{\nu\lambda}} + \frac{1}{2}\bar{g}^{\mu\rho}\left[\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\lambda\nu}\right] \\
&= \bar{\Gamma}^{\mu}_{\nu\lambda} + \frac{1}{2}\bar{g}^{\mu\rho}\left[\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\lambda\nu} - 2h_{\rho\sigma}\bar{\Gamma}^{\sigma}_{\nu\lambda}\right].
\end{aligned}$$

b) [7 points] Consider a vector perturbation to a flat Fridmann-Robertson-Walker spacetime

$$\begin{aligned}
\bar{g}_{00} &= -1, \\
\bar{g}_{i0} &= 0, \\
\bar{g}_{ij} &= a^2(t)\delta_{ij}, \\
h_{00} &= 0, \\
h_{i0} = h_{0i} &= a(t)G_i(t,\mathbf{x}), \\
h_{ij} &= 0,
\end{aligned}$$

where $\delta^{ij}\partial_j G_i = 0$ (it has zero divergence).

[1] In the project, when you used the dark matter and baryon velocity, v and v_b, they were assumed to only consist of this scalar part.

In the lectures you have already calculated the background evolution of this spacetime, and you can use these results here if you need them:

$$\begin{aligned}\bar{\Gamma}^0_{ij} &= \delta_{ij} a^2 H, \\ \bar{\Gamma}^i_{0j} = \bar{\Gamma}^i_{j0} &= \delta_{ij} H,\end{aligned}$$

the other unperturbed Christoffel symbols are zero.

Show that (compute four of the five symbols below for maximum marks)

$$\begin{aligned}\delta\Gamma^i_{00} &= \frac{\partial_0 h_{i0}}{a^2} \\ \delta\Gamma^0_{0i} = \delta\Gamma^0_{i0} &= H h_{0i} \\ \delta\Gamma^i_{j0} = \delta\Gamma^i_{0j} &= \frac{\partial_j h_{i0} - \partial_i h_{0j}}{2a^2} \\ \delta\Gamma^0_{ij} &= -\frac{\partial_j h_{0i} + \partial_i h_{0j}}{2} \\ \delta\Gamma^i_{jk} &= -\delta_{jk} h_{i0} H.\end{aligned}$$

All the other perturbed Christoffel symbols are zero.

Solution:

$$\begin{aligned}\delta\Gamma^i_{00} &= \frac{1}{2}\bar{g}^{ii}\left[\partial_0 h_{i0} + \partial_0 h_{i0} - \partial_i \underbrace{h_{00}}_{=0} - 2h_{i0}\underbrace{\bar{\Gamma}^0_{00}}_{=0}\right] \\ &= \frac{1}{2}\frac{1}{a^2}\left[2\partial_0 h_{i0}\right] \\ &= \frac{\partial_0 h_{i0}}{a^2}. \\ \delta\Gamma^0_{0i} = \delta\Gamma^0_{i0} &= \frac{1}{2}\bar{g}^{00}\left[\partial_0 h_{0i} + \partial_i \underbrace{h_{00}}_{=0} - \partial_0 h_{i0} - 2h_{0j}\bar{\Gamma}^j_{i0}\right] \\ &= \frac{1}{2}(-1)\left[-2h_{0j}\delta_{ij}H\right] \\ &= H h_{0i}. \\ \delta\Gamma^i_{j0} = \delta\Gamma^i_{0j} &= \frac{1}{2}\bar{g}^{ii}\left[\partial_j h_{i0} + \partial_0 \underbrace{h_{ij}}_{=0} - \partial_i h_{0j} - 2h_{i0}\underbrace{\bar{\Gamma}^0_{0j}}_{=0}\right] \\ &= \frac{1}{2}\frac{1}{a^2}\left[\partial_j h_{i0} - \partial_i h_{0j}\right] \\ &= \frac{\partial_j h_{i0} - \partial_i h_{0j}}{2a^2}. \\ \delta\Gamma^0_{ij} &= \frac{1}{2}\bar{g}^{00}\left[\partial_j h_{0i} + \partial_i h_{0j} - \partial_0 \underbrace{h_{ij}}_{=0} - 2h_{0k}\underbrace{\bar{\Gamma}^k_{ij}}_{=0}\right] \\ &= \frac{1}{2}(-1)\left[\partial_j h_{0i} + \partial_i h_{0j}\right] \\ &= \frac{-\partial_j h_{0i} - \partial_i h_{0j}}{2}. \\ \delta\Gamma^i_{jk} &= \frac{1}{2}\bar{g}^{ii}\left[\partial_k \underbrace{h_{ij}}_{=0} + \partial_j \underbrace{h_{ki}}_{=0} - \partial_i \underbrace{h_{jk}}_{=0} - 2h_{i0}\bar{\Gamma}^0_{jk}\right] \\ &= \frac{1}{2}\frac{1}{a^2}\left[-2h_{i0}\delta_{jk}a^2 H\right] \\ &= -\delta_{jk} h_{i0} H.\end{aligned}$$

c) [7 points] The perturbed Ricci tensor is then given by:

$$\delta R_{\mu\nu} = \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} - \partial_\nu \delta\Gamma^\alpha_{\mu\alpha} + \delta\Gamma^\beta_{\mu\nu}\bar{\Gamma}^\alpha_{\alpha\beta} + \delta\Gamma^\alpha_{\alpha\beta}\bar{\Gamma}^\beta_{\mu\nu} - \delta\Gamma^\beta_{\mu\alpha}\bar{\Gamma}^\alpha_{\nu\beta} - \delta\Gamma^\alpha_{\nu\beta}\bar{\Gamma}^\beta_{\mu\alpha}.$$

Show that

$$\delta R_{i0} = h_{0i}\left(\frac{\ddot{a}}{a} + 2H^2\right) - \frac{\nabla^2 h_{i0}}{2a^2}.$$

Hint: Remember that $\delta^{ij}\partial_j G_i = 0$.

Solution: Let's take each term in turn:

$$\begin{aligned}
\partial_\alpha \delta\Gamma^\alpha_{i0} &= \partial_0 \delta\Gamma^0_{i0} + \partial_j \delta\Gamma^j_{i0} \\
&= \partial_0(Hh_{0i}) + \frac{1}{2a^2}(\partial_i \underbrace{\partial_j h_{j0}}_{=0} - \nabla^2 h_{i0}) \\
&= \partial_0(Hh_{0i}) - \frac{\nabla^2 h_{i0}}{2a^2}. \\
\partial_0 \delta\Gamma^\alpha_{i\alpha} &= \partial_0 \delta\Gamma^0_{i0} - \partial_0 \delta\Gamma^j_{ij} \\
&= \partial_0(Hh_{0i}) - \partial_0(Hh_{i0}) \\
&= 0. \\
\delta\Gamma^\beta_{i0}\bar{\Gamma}^\alpha_{\alpha\beta} &= \delta\Gamma^0_{i0}\underbrace{\bar{\Gamma}^j_{j0}}_{=3H} + \delta\Gamma^j_{i0}\underbrace{\bar{\Gamma}^\alpha_{\alpha 0}}_{=0} \\
&= 3H^2 h_{0i}. \\
\delta\Gamma^\alpha_{\alpha\beta}\bar{\Gamma}^\beta_{i0} &= \delta\Gamma^\alpha_{\alpha j}\bar{\Gamma}^j_{i0} \\
&= \delta\Gamma^\alpha_{\alpha i} H \\
&= \delta\Gamma^0_{0i} H + \delta\Gamma^j_{ji} H \\
&= H^2 h_{0i} - H^2 h_{i0} \\
&= 0. \\
\delta\Gamma^\beta_{i\alpha}\bar{\Gamma}^\alpha_{0\beta} &= H\delta\Gamma^j_{ij} \\
&= -H^2 h_{0i}. \\
\delta\Gamma^\alpha_{0\beta}\bar{\Gamma}^\beta_{i\alpha} &= \delta\Gamma^j_{00}\bar{\Gamma}^0_{ij} + \delta\Gamma^0_{0j}\bar{\Gamma}^j_{i0} \\
&= a^2 H \frac{\partial_0 h_{i0}}{a^2} + H^2 h_{0i} \\
&= H\partial_0 h_{i0} + H^2 h_{0i}.
\end{aligned}$$

Combining the terms we get

$$\begin{aligned}
\delta R_{i0} &= \partial_0(Hh_{0i}) - \frac{\nabla^2 h_{i0}}{2a^2} + 3H^2 h_{0i} + H^2 h_{0i} - H\partial_0 h_{i0} - H^2 h_{0i} \\
&= h_{0i}\partial_0 H + H\partial_0 h_{i0} - \frac{\nabla^2 h_{i0}}{2a^2} + 3H^2 h_{0i} - H\partial_0 h_{i0} \\
&= h_{0i}\left(\frac{\ddot{a}}{a} - H^2 + 3H^2\right) - \frac{\nabla^2 h_{i0}}{2a^2} \\
&= h_{0i}\left(\frac{\ddot{a}}{a} + 2H^2\right) - \frac{\nabla^2 h_{i0}}{2a^2}.
\end{aligned}$$

d) [5 points] Consider a vector perturbation to the energy momentum tensor

$$T_{\mu\nu} = \bar{p}g_{\mu\nu} + (\bar{\rho} + \bar{p})u_\mu u_\nu,$$

where $u_\mu = (-1, u_i)$, $\delta^{ij}\partial_j u_i = 0$ and u_i is small.

You can always rewrite Einstein equation as

$$R_{\mu\nu} = 8\pi G S_{\mu\nu},$$

where

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda}.$$

In this case, it is a very good idea to write the equation on this form, because it allows us to extract the $i0$'th component of the Einstein equation without having to calculate the Ricci scalar (which would take a considerable effort)!

Calculate δS_{i0}.

Solution:

$$\begin{aligned} S_{i0} &= \delta T_{i0} - \frac{1}{2} \underbrace{\bar{g}_{i0}}_{=0} \delta T^{\lambda}{}_{\lambda} - \frac{1}{2} h_{i0} \bar{T}^{\lambda}{}_{\lambda} \\ &= \bar{p} h_{i0} - (\bar{\rho} + \bar{p}) u_i - \frac{1}{2} h_{i0} (3\bar{p} - \rho) \\ &= \frac{1}{2} h_{i0} (\bar{\rho} - \bar{p}) - (\bar{\rho} + \bar{p}) u_i . \end{aligned}$$

Show that the corresponding component of the Einstein equation becomes:

$$\frac{1}{2} \nabla^2 G_i = 8\pi G (\bar{\rho} + \bar{p}) a u_i,$$

where $\nabla^2 \equiv \delta^{ij} \partial_i \partial_j$.

Hint: You can use the Friedmann equations for a flat spacetime to write $\bar{\rho}$ and $\bar{p}$ in terms of $a, \dot{a}, \ddot{a}$.

Solution: Let's start by noting that

$$\frac{1}{2}(\bar{\rho} - \bar{p}) = \frac{1}{2} \frac{1}{8\pi G} \left(3H^2 + 2\frac{\ddot{a}}{a} + H^2 \right) = \frac{1}{8\pi G} \left(\frac{\ddot{a}}{a} + 2H^2 \right).$$

The $i0$'th component of the Einstein equation then becomes

$$h_{0i} \left(\frac{\ddot{a}}{a} + 2H^2 \right) - \frac{\nabla^2 h_{i0}}{2a^2} = h_{i0} \left(\frac{\ddot{a}}{a} + 2H^2 \right) - 8\pi G (\bar{\rho} + \bar{p}) u_i,$$

giving us

$$\frac{1}{2} \nabla^2 G_i = 8\pi G (\bar{\rho} + \bar{p}) a u_i$$

e) [7 points] The momentum conservation equation $\nabla^\mu T_{\mu\nu} = 0$ for the vector perturbation takes the following form (you do not need to derive this):

$$\partial_0 \left[(\bar{\rho} + \bar{p}) u_i \right] + \frac{3\dot{a}}{a} \left[(\bar{\rho} + \bar{p}) u_i \right] = 0.$$

Let us define the vector density $V_i \equiv [(\bar{\rho} + \bar{p}) u_i]$. How do V_i and G_i evolve as functions of a as the universe expands? How is this evolution different for modes inside and outside the horizon?

Solution: We see that $V_i \propto 1/a^3$ and $G_i \propto 1/a^2$. This evolution is independent of k, so the modes inside and outside the horizon both decay at the same rate.

What does this mean for the impact of vector perturbations in cosmology? Contrast this with the case for scalar and tensor fluctuations.

Solution: Since the vector perturbations to the metric decay even for modes outside the horizon, any vector perturbations created in the very early universe these perturbations would quickly decay away. Scalar and Tensor perturbations, however, do not decay until (possibly) after they enter the horizon, so they are the ones that are relevant during recombination, and as seeds for structure formation.

This is the reason vector perturbations are not much studied and often barely mentioned in many textbooks.

Problem 6: Line of sight integration for neutrinos (AST9420) [14 points]

In the lectures we derived the line of sight integral for the photon multipoles at the present time. We started with the Boltzmann equation for the photon distribution $\Theta = \Theta(\eta, k, \mu)$ (here ignoring the quadrupole correction Π)

$$\frac{d\Theta}{d\eta} + ik\mu(\Theta + \Psi) = -\frac{d\Phi}{d\eta} - \frac{d\tau}{d\eta}(\Theta_0 - \Theta + i\mu v_b)$$

and showed that this implies the line of sight integral

$$\Theta_\ell(k,\eta_0) = \int_0^{\eta_0}\left[g(\Theta_0 + \Psi) + \left(\frac{d\Psi}{d\eta} - \frac{d\Phi}{d\eta}\right)e^{-\tau} - \frac{1}{k}\frac{d}{d\eta}(gv_b)\right]j_\ell(k\eta_0 - k\eta)d\eta$$

We are going to do this same exercise for neutrinos. The Boltzmann equation for the neutrino temperature perturbations $\mathcal{N} = \mathcal{N}(\eta, k, \mu)$ is given by

$$\frac{d\mathcal{N}}{d\eta} + ik\mu(\mathcal{N} + \Psi) = -\frac{d\Phi}{d\eta}$$

In this problem you can use that the multipole expansion is given by

$$\mathcal{N} = \sum_\ell \frac{2\ell+1}{i^\ell}\Theta_\ell P_\ell(\mu)$$

where $P_\ell(\mu)$ is the Legendre polynomials and the multipoles are given by

$$\mathcal{N}_\ell = i^\ell \int_{-1}^{1}\frac{d\mu}{2}\mathcal{N}P_\ell(\mu)$$

The spherical Bessel functions j_ℓ are defined by

$$j_\ell(x) = i^\ell \int_{-1}^{1} P_\ell(\mu)\frac{d\mu}{2}e^{-i\mu x}$$

a) [3 points] Integrate the equation for $\frac{d\mathcal{N}}{d\eta}$ to obtain an expression for $\mathcal{N}$ at the present time. State any approximations/assumptions you are making.

Solution: We can write this equation as

$$\frac{d}{d\eta}[(\mathcal{N} + \Psi)e^{ik\mu\eta}] = [\frac{d\Psi}{d\eta} - \frac{d\Phi}{d\eta}]e^{ik\mu\eta}$$

Integrating over 0 and η we find

$$[(\mathcal{N} + \Psi)e^{ik\mu(\eta-\eta_0)}]_0^{\eta_0} = \int [\frac{d\Psi}{d\eta} - \frac{d\Phi}{d\eta}]e^{ik\mu(\eta-\eta_0)}d\eta$$

Unlike for photons we don't have a $e^{-\tau}$ factor so we cannot ignore the boundary terms at $\eta = 0$ for that reason.

b) [5 points] Expand this result in multipoles $\mathcal{N}_\ell$ and show that for all $\ell > 0$ the result can be written on the form

$$\mathcal{N}_\ell(\eta = \eta_0, k) = \int_0^{\eta_0} S_\nu j_\ell(k\eta_0 - k\eta)d\eta$$

for some source-function $S_\nu = S_\nu(k,\eta)$. State any approximations/assumptions you are making. If needed you can assume the initial neutrino distribution only has a monopole. Hint: to simplify you can use the relation $f(\eta_*) = \int f(\eta)\delta(\eta - \eta_*)d\eta$ if needed.

Solution: Lets first deal with the boundary terms. On the left hand side we have $(\mathcal{N} + \Psi)_{\rm today} - (\mathcal{N} + \Psi)_{\rm ini}e^{-ik\mu\eta_0}$. When taking multipoles then $\Psi_{\rm today}$ will only contribute to the monopole so we can ignore this. For the second term we can use that at the initial time we only have a sizable monopole and quadrupole. Ignoring the quadrupole this terms will give us $-(\mathcal{N}_0 + \Psi)_{\rm ini}j_\ell(k\eta_0)$ when we take multipoles. Taking multipoles of the equation above we arrive at

$$\mathcal{N}_\ell = \int [(\mathcal{N}_0 + \Psi)\delta(\eta) + \frac{d\Psi}{d\eta} - \frac{d\Phi}{d\eta}]j_\ell(k\eta_0 - k\eta)d\eta$$

Thus it has the same form as for photons with $\delta(\eta)$ playing the role of the "visibility function" and the LSS is at the initial time $\eta = 0$ and $\tau = 0$.

c) [6 points] Compare your result to the known result for photons (given above) and discuss similarities and differences. Could you have obtained this result from that without doing any calculations? Give a physical interpretation of $\mathcal{N}_\ell$ just as we did for the photon multipoles in the lectures.

Solution: The evolution equations for the neutrino perturbations are identical to that of photons if we take the limit of the Thompson cross section going to zero. This limit means moving the LSS towards $\eta = 0$, taking $\tau \to 0$ and the visibility function then just becomes a delta-function at the initial time $g \to \delta(\eta)$ in the photon result. This procedure gives us the same result as we derived above. We are left with just the SW and an ISW term (without the $e^{-\tau}$ weighting). Since neutrinos have no interactions the perturbations are frozen in at the initial time and free-streams from there to us today, i.e. they are determined directly by the initial conditions set up by inflation. The only change to this comes from gravitational redshift effects: the SW and ISW effect. The ISW effect for neutrinos is sensitive to decay of the potentials at all times, not just from recombination till today due to the absence of the optical depth.

A bit more accurately neutrinos are in fact tightly coupled to the photons very early on which we have ignored here. If we assume that the neutrinos instantaneously decouple then we will get the same result with $\delta(\eta) \to \delta(\eta - \eta_{\rm decoupling})$. In any case the modes of interest today are outside the horizon at that time so this does not change much.

(What this means is that if we were able to measure the neutrino power-spectrum then it would have the same low-ℓ behavior as for photons (we will have a SW plateau with the late time ISW effect, but with no suppression from reionization so we could break the A_s-τ degeneracy). The rest of the spectrum is determined by the ISW effect (so no acoustic peaks). There will also be no damping tail so in principle the neutrino power-spectrum can be measured to very large ℓ and will contain information about the Hubble function (which determines when modes enter the horizon and starts decaying). This would be great for determining $N_{\rm eff}$, i.e. we would be able to tell if there are additional relativistic species beyond the standard model present in the early Universe.)