Home exam AST5220 / AST9420 Spring 2020

25 May 14:30 - 2 June 14:30

About the exam

There are six problems in this set. Five of the problems (1,2,3,4,5) are for AST5220 and five of the problems (2,3,4,5,6) are for AST9420. The maximum score you can get is 100 points.

Problem 1: Warm up (AST5220) [14 points]

This is a collection of some basic questions relating to things in this course. You should be brief, and answer with no more than 3-4 sentences per question.

a) [2 points] Why is the temperature of the cosmological neutrinos smaller than the temperature of the photons today?

b) [2 points] What is the equation of state for a perfect fluid? Mathematically what does it mean that the Universe is accelerating and what condition must the equation of state of a fluid responsible for an accelerated expansion of the Universe satisfy?

c) [2 points] What are the main reason(s) we solve the CMB equations in Fourier space instead of doing it in real space?

d) [2 points] A new physical effect is found that causes the theoretical CMB quadrupole (C_ℓ for $\ell = 2$) to increase by 1% while keeping everything else the same. Explain why would this almost be impossible to detect in a CMB experiment?

e) [2 points] What are the main advantages of line of sight integration over the traditional Boltzmann hierarchy approach?

f) [2 points] What is the visibility function and what is the physical interpretation of this? Why it is practically zero at very early times?

g) [2 points] In the Boltzmann equation for photons we encountered the term

$$\frac{\partial f}{\partial \hat{p}} \frac{d\hat{p}}{dt}$$

Explain why we can ignore this term when deriving an equation for the photon perturbation Θ. If this term could not be ignored then how would you compute $\frac{d\hat{p}}{dt}$? (Don't do the calculation)

Problem 2: The CMB and matter power-spectrum (AST5220/AST9420) [20 points]

This problem is about understanding features in the CMB and matter power-spectrum. You might need to do some very small calculations in a few questions, but its mainly about the physical understanding. Keep the answers brief.

a) [4 points] The line of sight integration expression for Θ_ℓ (ignoring quadrupolar corrections) is given by

$$\Theta_\ell(k,\eta_0) = \int_0^{\eta_0} \left[g(\Theta_0+\Psi) + \left(\frac{d\Psi}{d\eta} - \frac{d\Phi}{d\eta}\right)e^{-\tau} - \frac{1}{k}\frac{d}{d\eta}(gv_b)\right] j_\ell(k\eta_0 - k\eta)d\eta$$

Explain what the different terms above represent physically. Use what you know about the visibility function to write down an approximate expression for the integral of the first term. How do we explain this result physically, i.e. what does this tell us about how the anisotropies we observe today are formed?

b) [4 points] You are given an theoretical spectrum showing the Sachs-Wolfe plateau ($\ell < 30$) of the CMB power spectrum. C_ℓ is seen to increase as we go from large to small ℓ. What are the cosmological parameters and the related physical effects that can cause such a signal?

c) [4 points] Explain briefly the effects of changing the optical depth of reionizaiton on the CMB (including polarization) and matter power-spectrum. Give an example of a cosmological parameter the optical depth is degenerate with in the CMB power-spectrum.

d) [4 points] The evolution equation for tensor perturbation h are given by

$$\frac{d^2h}{d\eta^2} + 2\frac{1}{a}\frac{da}{d\eta}\frac{dh}{d\eta} + k^2h = 0$$

What kind of equation is this? The tensor perturbation h acts as a source for tensor perturbations in the photon distribution which adds to the scalar perturbation signal we computed in the lectures for the CMB power-spectrum. It only sources these only for scales where the amplitude h is sizable in the period around recombination. On the basis of how h evolves explain why we would not see any effects of tensor perturbations in the temperature power-spectrum for large ℓ.

e) [4 points] The matter power-spectrum today

$$P(k) = \Delta_M^2(a=1,k)P_{\rm priomordial}(k)$$

where $P_{\rm priomordial}(k) = \frac{2\pi^2}{k^3}A_s(k/k_{\rm pivot})^{n_s-1}$ is the primordial power-spectrum and $\Delta_M(a,k) \equiv \frac{\Phi(a,k)k^2}{4\pi G\bar{\rho}(a)a^2}$ is the matter density contrast has a peak around $k \sim 0.01h/{\rm Mpc}$. Explain the reason for why we get this peak based on how Δ_M grows in different regimes and give an expression for the wave-number $k_{\rm peak}$ it correspond to? Use this to discuss what cosmological parameter combination the position of the peak depends on (you can assume that the only relevant forms of energy in the Universe is matter and radiation and you don't need to derive how Δ_M evolves in different regimes - its enough to explain how it evolves).

Problem 3: Scattering processes involving baryons (AST5220/AST9420) [10 points]

When you derived the Boltzmann equation for baryons, you found one equation for the baryon density and one equation for the average baryon velocity. Give a short physical/intuitive explanation for what each of these two equations tell us.

Consider all these scattering processes involving baryons:

1. $p^+\gamma \to p^+\gamma$
2. $e^-\gamma \to e^-\gamma$
3. $e^-e^- \to e^-e^-$
4. $e^-e^+ \leftrightarrow \gamma\gamma$
5. $e^-p^+ \to e^-p^+$
6. $n \to e^-p^+\bar{\nu}_e$
7. $e^-\nu_e \to e^-\nu_e$

where p^+ denotes a proton, e^- an electron, e^+ a positron, n a neutron, ν_e an electron neutrino and γ a photon.

AST5220: For each of these scattering processes give a short physical/intuitive explanation for why that process would contribute or not contribute (or be negligible) to the collision term for the equation for baryon density.

AST9420: For each of these scattering processes give a short physical/intuitive explanation for why that process would contribute or not contribute (or be negligible) to the collision term for the equation for baryon density and/or the equation for baryon velocity.

Note that process number 4 can go in both directions, consider the process in each direction separately.

Problem 4: Recombination including Helium (AST5220/AST9420) [26 points]

In our treatment of recombination in the lectures we assumed there only was hydrogen in our Universe. However there is also a sizable amounts of helium in our Universe. Helium comes in the forms He^0, He^+ and He^{++} (i.e. neutral, singly ionized an doubly ionized helium) with corresponding masses m_{He^0}, m_{He^+} and $m_{\mathrm{He}^{++}}$. The mass ratio of helium to hydrogen that is generated in big bang nucleosynthesis is given by

$$Y_p \equiv \frac{4n_{\mathrm{He}}}{n_B}$$

where $n_{\mathrm{He}} = n^0_{\mathrm{He}} + n^+_{\mathrm{He}} + n^{++}_{\mathrm{He}}$. Likewise hydrogen comes in the two forms H^0 and H^+ and $n_{\mathrm{H}} = n_{\mathrm{H}^0} + n_{\mathrm{H}^+}$ with corresponding masses m_{H^0} and m_{H^+} and the total baryon number density is given by $n_B = 4n_{\mathrm{He}} + n_{\mathrm{H}}$ (which basically is the number density of protons/neutrons). In our Universe $Y_p \approx \frac{1}{4}$.

The interactions between helium, hydrogen and electrons relevant for recombination is the following:

$$\mathrm{He}^{++} + e^- \rightleftharpoons \mathrm{He}^+ + \gamma$$

$$\mathrm{He}^+ + e^- \rightleftharpoons \mathrm{He}^0 + \gamma$$

$$\mathrm{H}^+ + e^- \rightleftharpoons \mathrm{H}^0 + \gamma$$

You are in this problem going to derive a closed system of equations for recombination including both hydrogen and helium in the Saha approximation. You can use without proof that the equilibrium distribution $n_i^{\mathrm{Eq}} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-\frac{m_i - \mu_i}{T}}$ for a non-relativistic particle in the low temperature limit and that $g_e = 2$, $\frac{g_{\mathrm{He}^{++}}}{g_{\mathrm{He}^+}} = 2$, $\frac{g_{\mathrm{He}^+}}{g_{\mathrm{He}^0}} = 1$ and $\frac{g_{\mathrm{H}^+}}{g_{\mathrm{H}^0}} = \frac{1}{2}$.

a) [6 points] Write down the Saha equations for the three remaining interactions above. You are free to assume that the interactions are in chemical equilibrium such that $\mu_1 + \mu_2 = \mu_3 + \mu_4$ for a $1 + 2 \rightleftharpoons 3 + 4$ process. Simplify the equations you find where possible and explain the approximations you are making. The Saha equation for hydrogen was derived in the lectures and the result we found was

$$\frac{n_{\mathrm{H}^+} n_{\mathrm{e}^-}}{n_{\mathrm{H}^0}} = \frac{g_{\mathrm{H}^+} g_e}{g_{\mathrm{H}^0}} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

where ϵ_0 is the ionization energy of hydrogen.

b) [6 points] Introduce the ionization fractions $x_{\mathrm{He}^{++}} = \frac{n_{\mathrm{He}^{++}}}{n_{\mathrm{He}}}$, $x_{\mathrm{He}^+} = \frac{n_{\mathrm{He}^+}}{n_{\mathrm{He}}}$ and $x_{\mathrm{H}^+} = \frac{n_{\mathrm{H}^+}}{n_{\mathrm{H}}}$ and show that the system above can be written as a closed system for the three x_i's (i.e. the equations should only depend on T, n_{e^-}, the x_i's and physical constants). For hydrogen the result we derived in the lectures reads

$$\frac{x_{\mathrm{H}^+} n_{\mathrm{e}^-}}{1 - x_{\mathrm{H}^+}} = \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

c) [6 points] The final part is to obtain an expression for the electron number density. What is the number density of free electrons in terms of $n^+_{\mathrm{He}}, n^{++}_{\mathrm{He}}, n^+_{\mathrm{H}}$ and what is the assumption you are making to get this? Use this to express n_{e^-} in terms of n_B, i.e. $n_{\mathrm{e}^-} = F n_B$ for some F depending on the x_i's. Use this to express the system above in terms of n_B and F and write it on such a way that the left hand side depends only depends on the x_i's (and F) and the right hand side consists only of known quantities. As a check that you got the correct result show that when there is no helium in the Universe (i.e. $Y_p = 0$ and in this case $F = x_{\mathrm{H}^+}$) then one of the equations above reduces to the result we derived in the lectures

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_B} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\frac{\epsilon_0}{T}}$$

where $X_e = \frac{n_{\mathrm{e}^-}}{n_{\mathrm{H}}}$.

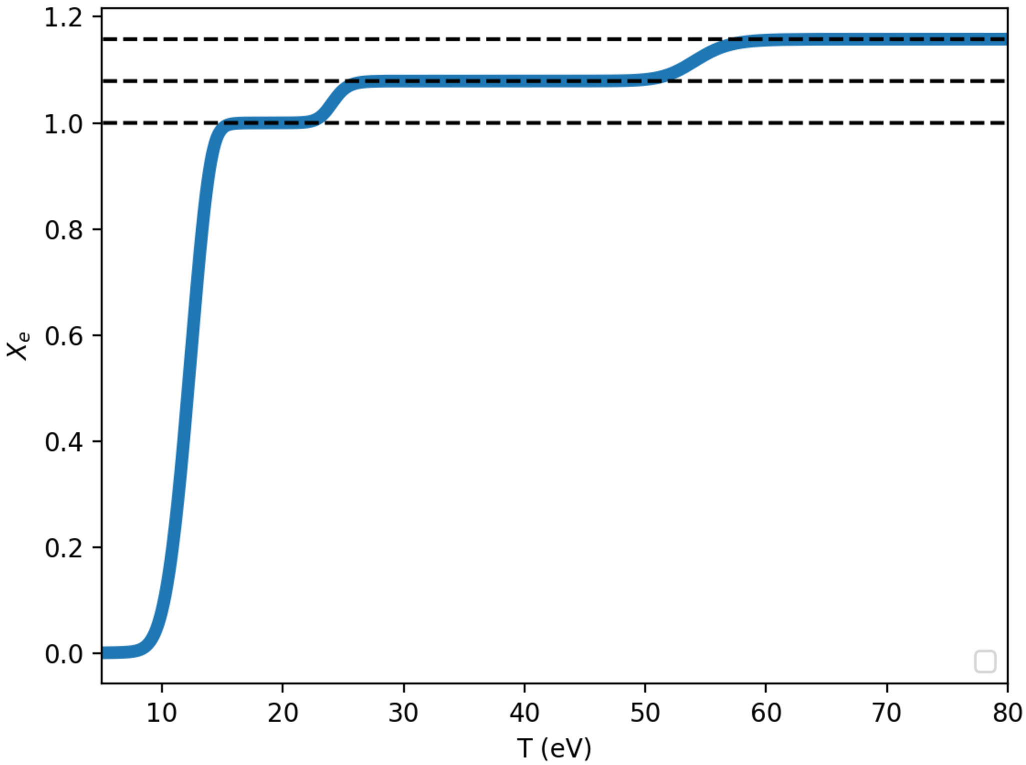

Figure 1: Evolution the free electron fraction as function of temperature. This is just a sketch and the numbers does not correspond to when the transitions happen in our Universe.

d) [8 points] What we are mainly interested in for recombination is the free electron fraction $X_e \equiv \frac{n_{e^-}}{n_H}$ as this determines the optical depth. In Figure 1 we show a sketch for the evolution of X_e as function of temperature. Explain from the equations you have derived (and give a physical reason for it) this evolution; what do the three regimes seen in the figure correspond to? If you did not manage the last problem try to explain the evolution on purely physical grounds.

Derive a formula for the value of X_e in terms of Y_p at the three plateaus you see (the three dashed lines). For this discussion you can use that $m_{\text{He}^{++}} + m_{\text{e}} - m_{\text{He}^+} \approx 25$ eV, $m_{\text{He}^+} + m_{\text{e}} - m_{\text{He}^0} \approx 55$ eV and $\epsilon_0 = m_{\text{H}^+} + m_{\text{e}} - m_{\text{H}^0} \approx 13$ eV. Hint: You are not meant to try to solve the equations exactly, but rather find approximate solutions for the x_i's in the different regimes either from the equations or from physical reasoning. Recall that if $T \ll T_*$ then $e^{-T_*/T} \approx 0$.

Problem 5: Vector perturbations (AST5220/AST9420) [30 points]

In the course you have studied both scalar and tensor perturbations, but now we will take a closer look at vector perturbations and how they evolve in an expanding universe.

A general 3D vector field, v_i, can always be decomposed into two parts, the part with no curl, and the part with no divergence:

$$v_i = v_i^{\parallel} + v_i^{\perp},$$

where $\nabla \times \mathbf{v}^{\parallel} = 0$ and $\nabla \cdot \mathbf{v}^{\perp} = 0$. The first part, $\mathbf{v}^{\parallel} = \nabla\phi$, can always be written as the gradient of some scalar field, ϕ and is therefore considered the scalar part of v_i[1]. The latter part of the velocity, $v_i^{\perp}$, is what is called the vector component of v_i and it is what we will consider in this problem.

We will only work with perturbations to first order in this problem, so you can safely neglect all higher order terms.

a) [4 points] Consider a general perturbed metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu}$ is a small perturbation and a $^{-}$ denotes an unperturbed quantity.

The Christoffel symbols are given by:

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left[\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\lambda\nu}\right].$$

Let us define the perturbed Christoffel symbols $\delta\Gamma^{\mu}_{\nu\lambda}$ by the following equation

$$\Gamma^{\mu}_{\nu\lambda} = \bar{\Gamma}^{\mu}_{\nu\lambda} + \delta\Gamma^{\mu}_{\nu\lambda}.$$

Show that to first order in the perturbation $h_{\mu\nu}$

$$\delta\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} \bar{g}^{\mu\rho} \left[\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\lambda\nu} - 2 h_{\rho\sigma} \bar{\Gamma}^{\sigma}_{\nu\lambda}\right].$$

Hint: $h^{\mu\nu} \equiv g^{\mu\nu} - \bar{g}^{\mu\nu} = -\bar{g}^{\mu\rho} h_{\rho\sigma} \bar{g}^{\sigma\nu} \neq \bar{g}^{\mu\rho} h_{\rho\sigma} \bar{g}^{\sigma\nu}$.

b) [7 points] Consider a vector perturbation to a flat Fridmann-Robertson-Walker spacetime

$$\begin{aligned}
\bar{g}_{00} &= -1, \\
\bar{g}_{i0} &= 0, \\
\bar{g}_{ij} &= a^2(t)\delta_{ij}, \\
h_{00} &= 0, \\
h_{i0} = h_{0i} &= a(t) G_i(t, \mathbf{x}), \\
h_{ij} &= 0,
\end{aligned}$$

where $\delta^{ij}\partial_j G_i = 0$ (it has zero divergence).

In the lectures you have already calculated the background evolution of this spacetime, and you can use these results here if you need them:

$$\begin{aligned}
\bar{\Gamma}^0_{ij} &= \delta_{ij} a^2 H, \\
\bar{\Gamma}^i_{0j} = \bar{\Gamma}^i_{j0} &= \delta_{ij} H,
\end{aligned}$$

the other unperturbed Christoffel symbols are zero.

Show that (compute four of the five symbols below for maximum marks)

$$\begin{aligned}
\delta\Gamma^i_{00} &= \frac{\partial_0 h_{i0}}{a^2} \\
\delta\Gamma^0_{0i} = \delta\Gamma^0_{i0} &= H h_{0i} \\
\delta\Gamma^i_{j0} = \delta\Gamma^i_{0j} &= \frac{\partial_j h_{i0} - \partial_i h_{0j}}{2a^2} \\
\delta\Gamma^0_{ij} &= -\frac{\partial_j h_{0i} + \partial_i h_{0j}}{2} \\
\delta\Gamma^i_{jk} &= -\delta_{jk} h_{i0} H.
\end{aligned}$$

[1] In the project, when you used the dark matter and baryon velocity, v and v_b, they were assumed to only consist of this scalar part.

All the other perturbed Christoffel symbols are zero.

c) [7 points] The perturbed Ricci tensor is then given by:

$$\delta R_{\mu\nu} = \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} - \partial_\nu \delta\Gamma^\alpha_{\mu\alpha} + \delta\Gamma^\beta_{\mu\nu}\bar{\Gamma}^\alpha_{\alpha\beta} + \delta\Gamma^\alpha_{\alpha\beta}\bar{\Gamma}^\beta_{\mu\nu} - \delta\Gamma^\beta_{\mu\alpha}\bar{\Gamma}^\alpha_{\nu\beta} - \delta\Gamma^\alpha_{\nu\beta}\bar{\Gamma}^\beta_{\mu\alpha}.$$

Show that

$$\delta R_{i0} = h_{0i}\left(\frac{\ddot{a}}{a} + 2H^2\right) - \frac{\nabla^2 h_{i0}}{2a^2}.$$

Hint: Remember that $\delta^{ij}\partial_j G_i = 0$.

d) [5 points] Consider a vector perturbation to the energy momentum tensor

$$T_{\mu\nu} = \bar{p} g_{\mu\nu} + (\bar{\rho} + \bar{p}) u_\mu u_\nu,$$

where $u_\mu = (-1, u_i)$, $\delta^{ij}\partial_j u_i = 0$ and u_i is small.

You can always rewrite Einstein equation as

$$R_{\mu\nu} = 8\pi G S_{\mu\nu},$$

where

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda.$$

In this case, it is a very good idea to write the equation on this form, because it allows us to extract the $i0$'th component of the Einstein equation without having to calculate the Ricci scalar (which would take a considerable effort)!

Calculate δS_{i0}.

Show that the corresponding component of the Einstein equation becomes:

$$\frac{1}{2}\nabla^2 G_i = 8\pi G(\bar{\rho} + \bar{p}) a u_i,$$

where $\nabla^2 \equiv \delta^{ij}\partial_i\partial_j$.

Hint: You can use the Friedmann equations for a flat spacetime to write $\bar{\rho}$ and $\bar{p}$ in terms of $a, \dot{a}, \ddot{a}$.

e) [7 points] The momentum conservation equation $\nabla^\mu T_{\mu\nu} = 0$ for the vector perturbation takes the following form (you do not need to derive this):

$$\partial_0 \left[(\bar{\rho} + \bar{p}) u_i\right] + \frac{3\dot{a}}{a}\left[(\bar{\rho} + \bar{p}) u_i\right] = 0.$$

Let us define the vector density $V_i \equiv [(\bar{\rho} + \bar{p}) u_i]$. How do V_i and G_i evolve as functions of a as the universe expands? How is this evolution different for modes inside and outside the horizon?

What does this mean for the impact of vector perturbations in cosmology? Contrast this with the case for scalar and tensor fluctuations.

Problem 6: Line of sight integration for neutrinos (AST9420) [14 points]

In the lectures we derived the line of sight integral for the photon multipoles at the present time. We started with the Boltzmann equation for the photon distribution $\Theta = \Theta(\eta, k, \mu)$ (here ignoring the quadrupole correction Π)

$$\frac{d\Theta}{d\eta} + ik\mu(\Theta + \Psi) = -\frac{d\Phi}{d\eta} - \frac{d\tau}{d\eta}\left(\Theta_0 - \Theta + i\mu v_b\right)$$

and showed that this implies the line of sight integral

$$\Theta_\ell(k, \eta_0) = \int_0^{\eta_0} \left[g(\Theta_0 + \Psi) + \left(\frac{d\Psi}{d\eta} - \frac{d\Phi}{d\eta}\right)e^{-\tau} - \frac{1}{k}\frac{d}{d\eta}(gv_b)\right] j_\ell(k\eta_0 - k\eta)d\eta$$

We are going to do this same exercise for neutrinos. The Boltzmann equation for the neutrino temperature perturbations $\mathcal{N} = \mathcal{N}(\eta, k, \mu)$ is given by

$$\frac{d\mathcal{N}}{d\eta} + ik\mu(\mathcal{N} + \Psi) = -\frac{d\Phi}{d\eta}$$

In this problem you can use that the multipole expansion is given by

$$\mathcal{N} = \sum_\ell \frac{2\ell + 1}{i^\ell}\Theta_\ell P_\ell(\mu)$$

where $P_\ell(\mu)$ is the Legendre polynomials and the multipoles are given by

$$\mathcal{N}_\ell = i^\ell \int_{-1}^{1} \frac{d\mu}{2}\mathcal{N} P_\ell(\mu)$$

The spherical Bessel functions j_ℓ are defined by

$$j_\ell(x) = i^\ell \int_{-1}^{1} P_\ell(\mu)\frac{d\mu}{2}e^{-i\mu x}$$

a) [3 points] Integrate the equation for $\frac{d\mathcal{N}}{d\eta}$ to obtain an expression for $\mathcal{N}$ at the present time. State any approximations/assumptions you are making.

b) [5 points] Expand this result in multipoles $\mathcal{N}_\ell$ and show that for all $\ell > 0$ the result can be written on the form

$$\mathcal{N}_\ell(\eta = \eta_0, k) = \int_0^{\eta_0} S_\nu j_\ell(k\eta_0 - k\eta)d\eta$$

for some source-function $S_\nu = S_\nu(k, \eta)$. State any approximations/assumptions you are making. If needed you can assume the initial neutrino distribution only has a monopole. Hint: to simplify you can use the relation $f(\eta_*) = \int f(\eta)\delta(\eta - \eta_*)d\eta$ if needed.

c) [6 points] Compare your result to the known result for photons (given above) and discuss similarities and differences. Could you have obtained this result from that without doing any calculations? Give a physical interpretation of $\mathcal{N}_\ell$ just as we did for the photon multipoles in the lectures.