UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Postponed exam for AST5220 — Cosmology II

Date: Thursday, August 16th, 2018

Time: 14.30 – 18.30

The exam set consists of 11 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions [20 p]

Answer each question with one to four sentences.

a) How can we write the spatial cold dark matter overdensity, $\delta(\vec{x}, \eta) \equiv (\rho(\vec{x}, \eta) - \rho^{(0)}(\eta))/\rho^{(0)}(\eta)$, in terms of the transfer function, $T_\delta(k, \eta)$ (this is the quantity that we called $\delta(k, \eta)$ in the project, and solved for in milestone III), and the initial condition for the potential, $\Phi(\vec{k}, \eta_{\text{init}})$? [4p]

Solution: In order to get the Fourier coefficients of the overdensity for the particular initial conditions $\Phi(\vec{k}, \eta_{\text{init}})$, we just multiply the initial conditions with the transfer function:

$$\delta(\vec{k}, \eta) = T_\delta(k, \eta)\Phi(\vec{k}, \eta_{\text{init}}).$$

In order to get the spatial overdensity we need to take the Fourier transform of this:

$$\delta(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^3} T_\delta(k, \eta)\Phi(\vec{k}, \eta_{\text{init}})e^{i\vec{k}\cdot\vec{x}}.$$

b) Why is it useful to use the line-of-sight integration method when finding the Cosmic Microwave Bacground (CMB) temperature today from the Einstein-Boltzmann equations? [4p]

Solution: The main advantage is computational speed. By formally integrating the Boltzmann equations before expanding into multipole moments, as opposed to expanding before integrating, one only has to solve for ≈ 6 photon multipole moments rather than l_{max} moments, even when calculating the full CMB spectrum to high l's. A second advantage is a clearer physical interpretation of the various effects that impact the spectrum, such as Sachs-Wolfe, Doppler, Integrated Sachs-Wolfe etc.

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0. \tag{1}$$

c) Eq. 1 is called the *geodesic* equation. Explain what the different terms mean, what the equation describes, and when it is useful in cosmology. [4p]

Solution: The geodesic equation describes locally flat curves in spacetime, and it is the equation of motion for a point particle in spacetime. The first term, $\frac{d^2x^\mu}{d\lambda^2}$, is the four-acceleration of the particle. The Christoffel symbols, $\Gamma^\mu{}_{\alpha\beta}$, incorporates both the effect that the curvature of spacetime has on the particles and also the effects of the coordinate system you are using to describe the curves. $\frac{dx^\mu}{d\lambda}$ is the four-momentum of the particle. The geodesic equation is useful in cosmology when we want to know how a particle evolves in time, as for example when we are deriving the Boltzmann equation for that particle species.

d) Which particle species, and what interactions among them, are relevant for the formation of the CMB? [4p]

Solution: The most important particles are the photons, electrons, protons, dark matter (and neutrinos). The protons and electrons are strongly coupled together by Coloumb interaction, behave as a single fluid, and later are bound together in neutral hydrogen. The photons are coupled to free electrons through Thomson scattering, and this determines the time of last scattering of the photons. All the particles interact through the gravitational force.

e) What is a tensor, and why are tensors useful in physics? [4p]

Solution: A tensor is a quantity that transforms in a specific way under coordinate transformations. We want the laws of physics to be independent of the coordinate system that we use to describe it, so we express the physical equations as tensor equations to make sure that they are manifestly the same in every coordinate system.

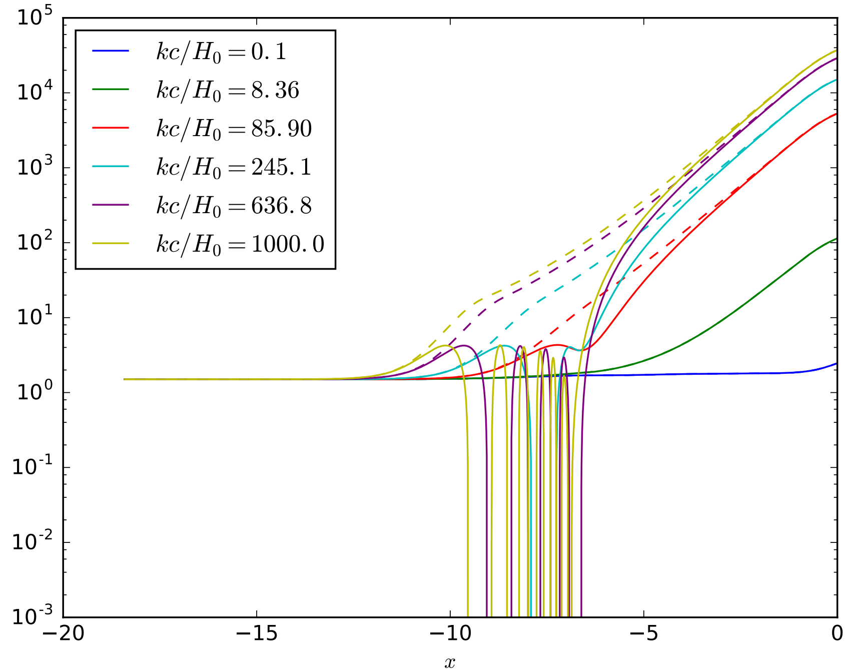

Figure 1: Plot of some quantities as function of time ($x \equiv \ln a$).

Problem 2 – Physical intuition [20 p]

Describe what Fig. 1 shows us.
Hint: Here are some questions that might be helpful:

- What quantities are plotted in Fig. 1?

- Describe the different epochs (of time) visible in the figure, and what the epochs correspond to physically.

- Explain the shape of the solid curves and the dotted curves and why they are different.

- Explain why the curves corresponding to different values of k behave differently.

Problem 3 – Boltzmann Equation for Number Density, Recombination [30 p]

a) Write down the general form of the collisionless Boltzmann equation (write out all the terms). Explain what the equation describes and what all the quantities that go into the equation represent physically. [8p]

b) For the rest of Problem 3, assume that the universe is homogeneous, isotropic and flat with metric $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$.

The number density of species i is given by:

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i. \tag{2}$$

- Derive the collisionless Boltzmann equation for the number density of a generic particle species in this case (you do not need to solve it, just derive the equation).

Hint: For a homogeneous, isotropic and flat universe, the only non-zero Christoffel symbols are (you do not need to show this!):

$$\Gamma^0{}_{ij} = \delta_{ij} a^2 H, \tag{3}$$

$$\Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \delta_{ij} H, \tag{4}$$

where $H = (da/dt)/a$ is the Hubble constant.

Hint: The zero'th component of the geodesic equation (Eq. 1) might be useful.

Hint: You can use the relation $dp/dE = E/p$ in order to write the equation in terms of p if, for some reason, you find yourself wanting to take an integral over p. [8p]

Solution: Since the universe is homogeneous and isotropic, the distribution function, f, can only be a function of $P^0 = E$, and t. The BE becomes:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial E}\frac{dE}{dt} = 0.$$

To find dE/dt we use the geodesic eq:

$$\begin{aligned}
\frac{dE}{dt} &= \frac{dE}{d\lambda}\frac{d\lambda}{dt} = \frac{1}{E}\frac{dE}{d\lambda}, \\
&= -\frac{1}{E}\Gamma^0{}_{\alpha\beta}P^\alpha P^\beta \\
&= -\frac{1}{E}\Gamma^0{}_{ij}P^i P^j \\
&= -\frac{H}{E}\underbrace{a^2\delta_{ij}P^i P^j}_{=p^2} \\
&= -H\frac{p^2}{E}.
\end{aligned}$$

Writing the equation in terms of p instead of E we get:

$$\frac{\partial f}{\partial t} - Hp\frac{\partial f}{\partial p} = 0.$$

Integrating over $g_i \int \frac{d^3p}{(2\pi)^3}$ we get:

$$\frac{dn}{dt} - Hg_i \int \frac{d^3p}{(2\pi)^3} p\frac{\partial f}{\partial p} = 0,$$
$$\frac{dn}{dt} + Hg_i 4\pi \int \frac{dp}{(2\pi)^3} \frac{\partial p^3}{\partial p} f = 0,$$
$$\frac{dn}{dt} + 3Hn = 0,$$

Where we used integration by parts to move the partial derivative away from f.

c) For the process of recombination, the collision process $e^-p^+ \to H\gamma$, where e^- is an electron, p^+ is a proton, H is a neutral hydrogen atom and γ is a photon, needs to be included. In that case we can rewrite the Boltzmann equation in terms of the free electron fraction $X_e \equiv n_e/n_b = n_p/n_b$, where $n_b = n_p + n_H$ is the total amount of protons (we are neglecting Helium and other heavier elements), into the Peebles equation (you dont have to do this!):

$$\frac{dX_e}{dt} = \left[(1 - X_e)\beta - X_e^2 n_b \alpha^{(2)}\right], \tag{5}$$

where

$$\beta \equiv \langle \sigma v \rangle \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T},$$

and

$$\alpha^{(2)} \equiv \langle \sigma v \rangle.$$

- What do the two terms on the right hand side correspond to physically? Do the signs make sense? What about the prefactors $(1 - X_e)$ and X_e^2? [8p]

d) Explain qualitatively the process of recombination. Also, why is it relevant for the formation of the CMB? [6p]

Problem 4 – Cold Dark Matter [30 p]

In the lectures (and in the book) we derived evolution equation for the cold dark matter (CDM) overdensities, $\delta(\vec{x},t)$, and average velocity, $v^i(\vec{x},t)$, using the Boltzmann equation for CDM. Today, however, you are going to use the fact that CDM can be treated as a perfect fluid to derive the same evolution equations from the conservation equation

$$\nabla_\mu T^\mu{}_\nu = 0. \tag{6}$$

The energy momentum tensor of a perfect fluid is determined by two scalar quantities, the energy density, $\rho(\vec{x},t)$, and the pressure, $P(\vec{x},t)$:

$$T^\mu{}_\nu = (\rho + P)U^\mu U_\nu + \delta^\mu{}_\nu P, \tag{7}$$

where U^μ is the four-velocity of the fluid.

a) For CDM we usually neglect the pressure, P. Why is this a good approximation? [5p]

Neglecting the pressure gives us:

$$T^\mu{}_\nu = \rho U^\mu U_\nu, \tag{8}$$

where $\rho(\vec{x},t)$ is the energy density and $U^\mu(\vec{x},t)$ is the four-velocity of the CDM fluid.

It is often useful to separate the energy momentum tensor into a zero'th order part and a perturbed part:

$$T^\mu{}_\nu(\vec{x},t) = T^{(0)\mu}{}_\nu(t) + \delta T^\mu{}_\nu(\vec{x},t). \tag{9}$$

We will be working in the Newtonian gauge, given by:

$$ds^2 = -[1 + 2\Psi(\vec{x},t)]dt^2 + a^2(t)[1 + 2\Phi(\vec{x},t)]\delta_{ij}dx^i dx^j. \tag{10}$$

b) Use the invariant product of the four-momentum, $P^\mu \equiv dx^\mu/d\lambda$, with itself

$$g_{\mu\nu}P^\mu P^\nu = -m^2, \tag{11}$$

together with the definitions of the squares of the physical momentum, p^i and physical energy, E

$$p^2 \equiv g_{ij}P^i P^j, \tag{12}$$
$$E^2 \equiv p^2 + m^2, \tag{13}$$

to show that the four-momentum is given (to first order) by

$$P^\mu = \left((1-\Psi)E, \frac{(1-\Phi)}{a}p^i\right). \tag{14}$$

[7p]

Solution: We have

$$g_{\mu\nu}P^\mu P^\nu = \underbrace{g_{00}P^0P^0}_{-E^2} + \underbrace{g_{ij}P^iP^j}_{p^2} = -m^2.$$

This gives us

$$
\begin{aligned}
P^0 &= \sqrt{\frac{-1}{g_{00}}}E \\
&= (1-\Psi)E,
\end{aligned}
$$

and

$$
\begin{aligned}
P^i &= \sqrt{\frac{1}{a^2(1+2\Phi)}}p^i \\
&= \frac{(1-\Phi)}{a}p^i.
\end{aligned}
$$

Using this, we can show that for CDM, the four velocity, $U^\mu \equiv dx^\mu/d\tau = P^\mu/m$, is given by (you do not need to show this!)

$$
U^\mu = \left((1-\Psi), \frac{(1-\Phi)}{a}v^i\right), \tag{15}
$$

and that

$$
U_\mu = \left(-(1+\Psi), a(1+\Phi)v^i\right). \tag{16}
$$

The four-velocity of the CDM fluid is given by Eqs 15 and 16, only that v^i is then the velocity of the fluid (i.e. the average velocity of the particles), which we assume to be small, and not the velocity of any individual particle. Defining the CDM overdensity

$$
\delta(\vec{x},t) \equiv \frac{\rho(\vec{x},t)-\rho^{(0)}(t)}{\rho^{(0)}(t)}, \tag{17}
$$

we can show that, to first order, the perturbed part of the CDM energy momentum tensor is given by (you do not need to show this!)

$$
\delta T^\mu{}_\nu = \rho^{(0)} \begin{pmatrix} -\delta & av^1 & av^2 & av^3 \\ -v^1/a & 0 & 0 & 0 \\ -v^2/a & 0 & 0 & 0 \\ -v^3/a & 0 & 0 & 0 \end{pmatrix}. \tag{18}
$$

We can then use the time component of the conservation equation

$$
\nabla_\mu T^\mu{}_0 = 0 \tag{19}
$$

to derive the evolution equations for the CDM density (you do not have to do this!):

$$
\frac{d\rho^{(0)}}{dt} + 3H\rho^{(0)} = 0, \tag{20}
$$

$$
\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0. \tag{21}
$$

c) Use the spatial components of the conservation equation

$$\nabla_\mu T^\mu{}_i = 0 \tag{22}$$

to derive the evolution equations for the CDM velocity.

You can use the following result (without deriving it):

$$\nabla_\mu T^{(0)\mu}{}_i = \rho^{(0)} \frac{\partial \Psi}{\partial x^i}. \tag{23}$$

Remember also that the covariant derivative of $\delta T^\mu{}_i$ is given by:

$$\nabla_\mu \delta T^\mu{}_i = \frac{\partial \delta T^\mu{}_i}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu} \delta T^\alpha{}_i - \Gamma^\alpha{}_{\mu i} \delta T^\mu{}_\alpha.$$

Show that you get the following

$$\frac{\partial v^i}{\partial t} + H v^i + \frac{1}{a}\frac{\partial \Psi}{\partial x^i} = 0. \tag{24}$$

Hint 1: Remember that you can drop any terms second order in the small quantities. Do this as early as possible!

Hint 2: If you use the hints from Problem 3 b), you should not have to calculate any new Christoffel symbols. [8p]

Solution:

$$\begin{aligned}
\nabla_\mu \delta T^\mu{}_i &= \frac{\partial \delta T^\mu{}_i}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu} \delta T^\alpha{}_i - \Gamma^\alpha{}_{\mu i} \delta T^\mu{}_\alpha \\
&= \frac{\partial \delta T^0{}_i}{\partial t} + \frac{\partial}{\partial x^j} \underbrace{\delta T^j{}_i}_{=0} + \Gamma^\mu{}_{0\mu} \delta T^0{}_i - \Gamma^0{}_{ji} \delta T^j{}_0 - \Gamma^j{}_{0i} \delta T^0{}_j \\
&= \frac{\partial}{\partial t}[\rho^{(0)} a v^i] + 3H[\rho^{(0)} a v^i] \underbrace{- a^2 H \left(-\rho^{(0)} \frac{v^i}{a} \right) - H[\rho^{(0)} a v^i]}_{=0} \\
&= a v^i \underbrace{\left(\frac{d\rho^{(0)}}{dt} + 3H\rho^{(0)} \right)}_{=0} + a\rho^{(0)} \left(H v^i + \frac{\partial v^i}{\partial t} \right).
\end{aligned}$$

We then get:

$$\nabla_\mu T^\mu{}_i = a\rho^{(0)} \left(\frac{\partial v^i}{\partial t} + H v^i + \frac{1}{a}\frac{\partial \Psi}{\partial x^i} \right) = 0,$$

or

$$\frac{\partial v^i}{\partial t} + H v^i + \frac{1}{a}\frac{\partial \Psi}{\partial x^i} = 0.$$

d) Explain qualitatively one way in which observations of the CMB provides evidence for the existence of CDM. [10p]

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{25}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{26}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{27}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.
- For a perfect fluid, the energy-momentum tensor (in the rest frame of the fluid) is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{28}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{29}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{30}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{31}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{32}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{33}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{34}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{35}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{36}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{37}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{38}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{39}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{40}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{41}$$

1.4 Initial conditions

$$\Phi = 1 \tag{42}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{43}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{44}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{45}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{46}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{47}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{48}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{49}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{50}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{51}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{52}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{53}$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b}, \tag{54}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\text{K}/a$, and $\epsilon_0 = 13.605698\text{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \tag{55}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to1s} + \Lambda_\alpha}{\Lambda_{2s\to1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{56}$$

$$\Lambda_{2s\to1s} = 8.227\text{s}^{-1} \tag{57}$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{58}$$

$$n_{1s} = (1 - X_e) n_H \tag{59}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \tag{60}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b} \tag{61}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \tag{62}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0/T_b) \tag{63}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{64}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{65}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{66}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{67}$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{68}$$