UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam for AST5220/9420 — Cosmology II

Date: Monday, June 11th, 2018

Time: 09.00 – 13.00

The exam set consists of 13 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Note that AST5220 students are supposed to answer problems 1)-4), while AST9420 students answer problems 1)-3) and 5) Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions (AST5220 and AST9420) [20 p]

Answer each question with one to four sentences.

a) How is the spatial cold dark matter overdensity, $\delta(\vec{x},\eta) \equiv (\rho(\vec{x},\eta) - \rho^{(0)}(\eta))/\rho^{(0)}(\eta)$, found from the transfer function, $T_\delta(k,\eta)$ (this is the quantity that we called $\delta(k,\eta)$ in the project, and solved for in milestone III), and the initial condition for the potential, $\Phi(\vec{k},\eta_{\rm init})$? [4p]

Solution: In order to get the Fourier coefficients of the overdensity for the particular initial conditions $\Phi(\vec{k},\eta_{\rm init})$, we just multiply the initial conditions with the transfer function:

$$\delta(\vec{k},\eta) = T_\delta(k,\eta)\Phi(\vec{k},\eta_{\rm init}).$$

In order to get the spatial overdensity we need to take the Fourier transform of this:

$$\delta(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^3} T_\delta(k,\eta)\Phi(\vec{k},\eta_{\rm init})e^{i\vec{k}\cdot\vec{x}}.$$

b) Why is it simpler to solve the first order Einstein-Boltzmann equations in Fourier space rather than in position space? [4p]

Solution: Since the first order equations are linear, the different Fourier modes decouple, so you get a separate equation set for each mode $\vec{k}$. This turns the set of coupled partial differential equations into decoupled sets of oridinary differential equations, which are much easier to solve.

c) What is the main effect of photon diffusion on the Cosmic Microwave Bacground (CMB) temperature fluctuations? [4p]

Solution: Photon diffusion washes out perturbations in the photon fluid on small scales leading to an exponential suppression in the CMB angular power spectrum at high l's, corresponding to scales smaller than the photon diffusion length.

d) Which particle species, and what interactions among them, are relevant for the formation of the CMB? [4p]

Solution: The most important particles are the photons, electrons, protons, dark matter (and neutrinos). The protons and electrons are strongly coupled together by Coloumb interaction, behave as a single fluid, and later are bound together in neutral hydrogen. The photons are coupled to free electrons through Thomson scattering, and this determines the time of last scattering of the photons. All the particles interact through the gravitational force.

e) Explain qualitatively how inflation gave rise to the fluctuations that seeded the subsequent growth of structure in the universe? (Remember, only a short description is needed, max four sentences!) [4p]

Solution: Quantum fluctuations in the inflaton field makes inflation last slightly longer or slightly shorter at different points in space. This leads to classical perturbations in the energy density and pressure which serve as initial conditions for the subsequent growth of structure in the universe.

Problem 2 – Deriving the Friedmann Equation (AST5220 and AST9420) [20 p]

Consider a spatially flat Friedmann-Robertson-Walker spacetime with metric given by

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{1}$$

a) Calculate the Christoffel symbols ${\Gamma^0}_{ij}$, ${\Gamma^i}_{0j}$ and ${\Gamma^i}_{j0}$ for this spacetime. All other Christoffel symbols are equal to zero. Show that

$$\begin{aligned} {\Gamma^0}_{ij} &= \delta_{ij} a^2 H, \quad (2) \\ {\Gamma^i}_{0j} = {\Gamma^i}_{j0} &= \delta_{ij} H, \quad (3) \end{aligned}$$

where $H = (da/dt)/a$ is the Hubble constant. [6p]

Solution:

$$\begin{aligned} {\Gamma^0}_{ij} &= \frac{g^{0\nu}}{2}\left(g_{\nu i,j} + g_{\nu j,i} - g_{ij,\nu}\right) \\ &= \frac{g^{00}}{2}\left(g_{0i,j} + g_{0j,i} - g_{ij,0}\right) \\ &= \frac{(-1)}{2}\left(-\delta_{ij}\frac{d[a^2]}{dt}\right) \\ &= \delta_{ij} a^2 H. \end{aligned}$$

Since the Christoffel symbol is symmetric in it's two lower indices we get

$$\begin{aligned} {\Gamma^i}_{0j} \underbrace{=}_{{\Gamma^\mu}_{\alpha\beta} = {\Gamma^\mu}_{\beta\alpha}} {\Gamma^i}_{j0} &= \frac{g^{i\nu}}{2}\left(g_{\nu 0,j} + g_{\nu j,0} - g_{0j,\nu}\right) \\ &= \frac{g^{ii}}{2}\left(g_{i0,j} + g_{ij,0}\right) \\ &= \frac{\frac{1}{a^2}}{2}\left(\delta_{ij}\frac{d[a^2]}{dt}\right) \\ &= \delta_{ij} H. \end{aligned}$$

b) Show that:

$$R_{00} = -3\frac{d^2a/dt^2}{a}, \tag{4}$$

where $R_{\mu\nu}$ is the Ricci tensor. [6p]

Solution:

$$\begin{aligned}
R_{00} &= \underbrace{\Gamma^{\alpha}{}_{00,\alpha}}_{=0} - \Gamma^{\alpha}{}_{0\alpha,0} + \Gamma^{\alpha}{}_{\beta\alpha} \underbrace{\Gamma^{\beta}{}_{00}}_{=0} - \Gamma^{\alpha}{}_{\beta 0}\Gamma^{\beta}{}_{0\alpha} \\
&= -\Gamma^{i}{}_{0i,0} - \Gamma^{i}{}_{j0}\Gamma^{j}{}_{0i} \\
&= -\underbrace{\delta_{ii}}_{=3}\left(\frac{d^2a/dt^2}{a} - \frac{(da/dt)^2}{a^2}\right) - \underbrace{\delta_{ij}\delta_{ji}}_{=3}\frac{(da/dt)^2}{a^2} \\
&= -3\frac{d^2a/dt^2}{a}.
\end{aligned}$$

c) We also have (you do not need to show this):

$$R_{ij} = \delta_{ij}\left[2\left(\frac{da}{dt}\right)^2 + a\frac{d^2a}{dt^2}\right]. \tag{5}$$

Use the $_{00}$-component of the Einstein equation to derive the familiar Friedmann equation. You can assume that the matter in the universe can be described as a perfect fluid. [8p]

Solution: First calculate the Ricci scalar

$$\begin{aligned}
\mathcal{R} &= g^{00}R_{00} + g^{ij}R_{ij} \\
&= 3\frac{d^2a/dt^2}{a} + \frac{\delta_{ij}}{a^2}\left(\delta_{ij}\left[2\left(\frac{da}{dt}\right)^2 + a\frac{d^2a}{dt^2}\right]\right) \\
&= 3\frac{d^2a/dt^2}{a} + 6H^2 + 3\frac{d^2a/dt^2}{a} \\
&= 6\left(\frac{d^2a/dt^2}{a} + H^2\right).
\end{aligned}$$

Then the Einstein eq.

$$\begin{aligned}
R_{00} - \frac{g_{00}}{2}\mathcal{R} &= 8\pi G T_{00} \\
-3\frac{d^2a/dt^2}{a} + 3\left(\frac{d^2a/dt^2}{a} + H^2\right) &= 8\pi G g_{0\alpha}T^{\alpha}{}_{0} \\
3H^2 &= 8\pi G(-1)(-\rho).
\end{aligned}$$

which gives us the familiar Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\rho.$$

Problem 3 – Boltzmann Equation for Number Density (AST5220 and AST9420) [30 p]

For this problem, assume that the universe is homogeneous, isotropic and flat.

a) The collisionless Boltzmann equation for the number density of a generic particle species is given by

$$\frac{dn}{dt} + 3Hn = 0, \tag{6}$$

where n is the number density and $H = (da/dt)/a$ is the Hubble constant.

- What is the solution of this equation (it can be smart to solve the equation in terms of a instead of t)? What does the solution mean physically? [6p]

Solution: Let's write the equation in terms of a

$$\frac{dn}{dt} = \frac{dn}{da}\frac{da}{dt},$$

giving us

$$\frac{da}{dt}\left(\frac{dn}{da} + \frac{3}{a}n\right) = 0.$$

Since the quantity in the parenthesis needs to be zero we get

$$\frac{dn}{da} = -\frac{3}{a}n,$$

using a power-law ansatz ($n \propto a^k$) we see that

$$n = \frac{n_0}{a^3}.$$

Physically we see that, since there are no collisions, the total number of particles stays fixed, and the number density is then just diluted with the exapnded volume of the universe $n \propto 1/V$.

b) If we add a collision term, we can, under certain conditions, write the complete Boltzmann equation for the number density of species 1 on the form:

$$\frac{dn_1}{dt} + 3Hn_1 = n_1^{\rm eq} n_2^{\rm eq} \langle \sigma v \rangle \left[\frac{n_3 n_4}{n_3^{\rm eq} n_4^{\rm eq}} - \frac{n_1 n_2}{n_1^{\rm eq} n_2^{\rm eq}}\right], \tag{7}$$

where we have defined

$$n_i^{\rm eq} \equiv g_i \int \frac{d^3p}{(2\pi)^3} \exp\left(-E_i/T\right). \tag{8}$$

- What does this collision term represent physically?

Solution: The collision term takes into account collisions that can change the number density of species 1. In this case the collision takes into accout the processes $1, 2 \rightarrow 3, 4$ and the reverse. That is, a collision where a particle of species 1 meets a particle of species 2 and together they produce a particle of species 3 and a particle of species 4 (and the reverse).

- What do the different terms and factors on the right hand side of Eq. 7 mean physically. Do the signs of the two terms make sense?

Solution: $\langle \sigma v \rangle$ is the thermally averaged and velocity weigthed scattering cross section for the process $1,2 \to 3,4$, n_i is the number density of species i, while $n_i^{\rm eq}$ is the number density that species i would have if it was in thermal equilibrium (with zero chemical potential). The term on the right in the square bracket corresponds to the process $1,2 \to 3,4$, while the term on the left corresponds to the reverse process. It makes sense that the number of each of those collisions happening should be proportional to the number densities of the two species of particle that need to collide in each case. The signs also makes sense since the process $1,2 \to 3,4$ reduces the particle number of species 1, and the corresponding term comes with a minus sign. On the other hand, the process $3,4 \to 1,2$ increases the particle number of species 1, and thus comes with a positive sign.

- Qualitatively, how do the results from a) change when the collision term is introduced? [12p]

Solution: The addition of the collision term to the Boltzmann equation tends to drive the number density of the species towards thermal equilibrium. The solution will then depend on how important the collision term is compared to the "Hubble" term (the one taking into account the expansion of the universe). The relative importance of these terms can be found by comparing the scattering rate, $\Gamma \approx n_2 \langle \sigma v \rangle$, to the Hubble rate, H. If the Hubble rate is dominant, then the solution will look like the result in a), while if Γ is dominant the number density will follow the equilibrium value. The intermediate case is more complicated and it is where we need the full Boltzmann equation to find the solution.

c) The Saha equation corresponding to Eq. 7 is given by

$$\frac{n_3 n_4}{n_3^{\rm eq} n_4^{\rm eq}} = \frac{n_1 n_2}{n_1^{\rm eq} n_2^{\rm eq}} \tag{9}$$

- When is the Saha equation a good approximation? Explain! [4p]

Solution: The Saha equation is valid as long as the scattering rate $\Gamma \approx n_2 \langle \sigma v \rangle$ is much larger than the expansion rate, H. Since the scattering rate is multiplied by the term in the square bracket of Eq. 7, the term in the bracket needs to be very close to zero if Γ is really large, so this will lead to Eq. 9. The Saha is then simply the equation for the equilibrium number density of the species 1, 2, 3 and 4.

d) Derive the Saha equation for recombination (i.e. the process $e^- p^+ \to H\gamma$, where e^- is an electron, p^+ is a proton, H is a neutral hydrogen atom and γ is a photon). Write it in terms of the free electron fraction $X_e \equiv n_e/n_b = n_p/n_b$, where $n_b = n_p + n_H$ is the total amount of protons (we are neglecting Helium and other heavier elements).

Hint 1: You can use the fact that, for a non-relativistic particle species:

$$n_i^{\rm eq} = \left(\frac{m_i T}{2\pi} \right)^{3/2} e^{-m_i/T}.$$

Hint 2: For photons $n_\gamma = n_\gamma^{\rm eq}$. [8p]

Solution:

$$\frac{n_H n_\gamma}{n_H^{\mathbf{eq}} n_\gamma^{\mathbf{eq}}} = \frac{n_e n_p}{n_e^{\mathbf{eq}} n_p^{\mathbf{eq}}},$$
$$\frac{n_b(1-X_e)}{n_H^{\mathbf{eq}}} = \frac{n_b^2 X_e^2}{n_e^{\mathbf{eq}} n_p^{\mathbf{eq}}}.$$

where we have used the fact that the photons are in equilibrium. We get

$$\begin{aligned}
\frac{X_e^2}{(1-X_e)} &= \frac{1}{n_b}\frac{n_e^{\mathbf{eq}} n_p^{\mathbf{eq}}}{n_H^{\mathbf{eq}}} \\
&= \frac{1}{n_b}\left(\frac{m_e T}{2\pi}\right)^{3/2}\left(\frac{m_p T}{2\pi}\right)^{3/2}\left(\frac{m_H T}{2\pi}\right)^{-3/2} e^{-(m_e+m_p-m_H)/T} \\
&\approx \frac{1}{n_b}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\epsilon_0/T},
\end{aligned}$$

where we have assumed $m_p \approx m_H$ in the prefactor, and introduced the Hydrogen binding energy $\epsilon_0 = m_e + m_p - m_H$.

Problem 4 – Cold Dark Matter (AST5220) [30 p]

In the lectures (and in the book) we derived evolution equation for the cold dark matter (CDM) overdensities, $\delta(\vec{x},t)$, and average velocity, $v^i(\vec{x},t)$, using the Boltzmann equation for CDM. Today, however, you are going to use the fact that CDM can be treated as a perfect fluid to derive the same evolution equations from the conservation equation

$$\nabla_\mu T^\mu{}_\nu = 0. \tag{10}$$

The energy momentum tensor of the CDM fluid is given by

$$T^\mu{}_\nu = \rho U^\mu U_\nu, \tag{11}$$

where $\rho(\vec{x},t)$ is the energy density and $U^\mu(\vec{x},t)$ is the four-velocity of the CDM fluid.

It is often useful to separate the energy momentum tensor into a zero'th order part and a perturbed part:

$$T^\mu{}_\nu(\vec{x},t) = T^{(0)\mu}{}_\nu(t) + \delta T^\mu{}_\nu(\vec{x},t). \tag{12}$$

We will be working in the Newtonian gauge, given by:

$$ds^2 = -[1+2\Psi(\vec{x},t)]dt^2 + a^2(t)[1+2\Phi(\vec{x},t)]\delta_{ij}dx^i dx^j. \tag{13}$$

a) The four-momentum is given (to first order) by

$$P^\mu = \left((1-\Psi)E, \frac{(1-\Phi)}{a}p^i\right). \tag{14}$$

Using this, show that for CDM, the four velocity, $U^\mu \equiv dx^\mu/d\tau = P^\mu/m$, is given by

$$U^\mu = \left((1-\Psi), \frac{(1-\Phi)}{a}v^i\right), \tag{15}$$

and that

$$U_\mu = \left(-(1+\Psi), a(1+\Phi)v^i\right). \tag{16}$$

[3p]

Solution: We assume CDM to be non-relativistic, so (neglecting terms proportional to p^2/m^2) $E \approx m$ and $p^i \approx mv^i$. This gives us

$$U^\mu = P^\mu/m = \left((1-\Psi), \frac{(1-\Phi)}{a}v^i\right).$$

Furter we have

$$\begin{aligned} U_\mu = g_{\mu\nu}U^\nu &= \left(-(1+2\Psi)(1-\Psi), a^2(1+2\Phi)\frac{(1-\Phi)}{a}v^i\right) \\ &= \left(-(1+\Psi), a(1+\Phi)v^i\right). \end{aligned}$$

b) The four-velocity of the CDM fluid is given by Eqs 15 and 16, only that v^i is then the velocity of the fluid (i.e. the average velocity of the particles), which we assume to be small, and not the velocity of any individual particle. Defining the CDM overdensity

$$\delta(\vec{x},t) \equiv \frac{\rho(\vec{x},t) - \rho^{(0)}(t)}{\rho^{(0)}(t)}, \tag{17}$$

show that, to first order, the perturbed part of the CDM energy momentum tensor is given by

$$\delta T^{\mu}{}_{\nu} = \rho^{(0)} \begin{pmatrix} -\delta & av^1 & av^2 & av^3 \\ -v^1/a & 0 & 0 & 0 \\ -v^2/a & 0 & 0 & 0 \\ -v^3/a & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

[6p]

Solution: Let's start with $T^0{}_0$, we get

$$\begin{aligned} T^0{}_0 &= \rho^{(0)}(1+\delta)U^0U_0 \\ &= \rho^{(0)}(1+\delta)[-(1-\Psi)](1+\Psi) \\ &= -\rho^{(0)}(1+\delta) \\ &= \underbrace{-\rho^{(0)}}_{T^{(0)0}{}_0}\underbrace{-\rho^{(0)}\delta}_{\delta T^0{}_0}. \end{aligned}$$

We also get

$$\begin{aligned} T^0{}_i &= \rho^{(0)}(1+\delta)U^0U_i \\ &= \rho^{(0)}(1+\delta)(1-\Psi)[a(1+\Phi)v^i] \\ &= \rho^{(0)}av^i. \end{aligned}$$

and

$$\begin{aligned} T^i{}_0 &= \rho^{(0)}(1+\delta)U^iU_0 \\ &= \rho^{(0)}(1+\delta)\frac{(1-\Phi)}{a}v^i[-(1-\Psi)] \\ &= -\rho^{(0)}\frac{v^i}{a}. \end{aligned}$$

We also have

$$\begin{aligned} T^i{}_j &= \rho^{(0)}(1+\delta)U^iU_j \\ &\propto v^iv^j \\ &\approx 0. \end{aligned}$$

c) Use the time component of the conservation equation

$$\nabla_\mu T^\mu{}_0 = 0 \tag{19}$$

to derive the evolution equations for the CDM density.

You can use the following result (without deriving it):

$$\nabla_\mu T^{(0)\mu}{}_0 = -\frac{d\rho^{(0)}}{dt} - 3\rho^{(0)}\left(H + \frac{\partial \Phi}{\partial t}\right). \tag{20}$$

Remember also that the covariant derivative of $\delta T^\mu{}_0$ is given by:

$$\nabla_\mu \delta T^\mu{}_0 = \frac{\partial \delta T^\mu{}_0}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu}\delta T^\alpha{}_0 - \Gamma^\alpha{}_{\mu 0}\delta T^\mu{}_\alpha.$$

Separate the zero'th order equation from the first order one, and show that you get the following

$$\frac{d\rho^{(0)}}{dt} + 3H\rho^{(0)} = 0, \tag{21}$$

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0. \tag{22}$$

Hint 1: Remember that you can drop any terms second order in the small quantities. Do this as early as possible!

Hint 2: If you use the results from Problem 2 a), you should not have to calculate any new Christoffel symbols. [7p]

Solution: Let's start by finding $\nabla_\mu \delta T^\mu{}_0$. Since δT is already first order, we only need to consider the Christoffel symbols to zero'th order.

$$\begin{aligned}
\nabla_\mu \delta T^\mu{}_0 &= \frac{\partial \delta T^0{}_0}{\partial t} + \frac{\partial \delta T^i{}_0}{\partial x^i} + \Gamma^\mu{}_{\alpha\mu}\delta T^\alpha{}_0 - \Gamma^\alpha{}_{\mu 0}\delta T^\mu{}_\alpha \\
&= -\frac{d\rho^{(0)}}{dt}\delta - \rho^{(0)}\frac{\partial \delta}{\partial t} - \rho^{(0)}\frac{1}{a}\frac{\partial v^i}{\partial x^i} + \Gamma^i{}_{0i}\delta T^0{}_0 - \Gamma^i{}_{j0}\underbrace{\delta T^j{}_i}_{=0} \\
&= -\frac{d\rho^{(0)}}{dt}\delta - \rho^{(0)}\frac{\partial \delta}{\partial t} - \rho^{(0)}\frac{1}{a}\frac{\partial v^i}{\partial x^i} - 3H\rho^{(0)}\delta
\end{aligned}$$

putting it all together we get

$$\nabla_\mu T^\mu{}_0 = -\frac{d\rho^{(0)}}{dt} - 3\rho^{(0)}\left(H + \frac{\partial \Phi}{\partial t}\right) - \frac{d\rho^{(0)}}{dt}\delta - \rho^{(0)}\frac{\partial \delta}{\partial t} - \rho^{(0)}\frac{1}{a}\frac{\partial v^i}{\partial x^i} - 3H\rho^{(0)}\delta = 0.$$

The zero'th order equation is then

$$\frac{d\rho^{(0)}}{dt} + 3\rho^{(0)}H = 0.$$

And the first order equation is

$$-\rho^{(0)}\left(\frac{\partial\delta}{\partial t}+\frac{1}{a}\frac{\partial v^i}{\partial x^i}+3\frac{\partial\Phi}{\partial t}\right)-\delta\underbrace{\left(\frac{d\rho^{(0)}}{dt}+3H\rho^{(0)}\right)}_{=0}=0.$$

which gives us

$$\frac{\partial\delta}{\partial t}+\frac{1}{a}\frac{\partial v^i}{\partial x^i}+3\frac{\partial\Phi}{\partial t}=0.$$

d) Using the spatial components of the conservation equation, we can also derive an equation for the CDM velocity (you do not have to do this!). Give a physical/intuitive explanation of the different terms in each of the three equations:

$$\frac{d\rho^{(0)}}{dt}+3H\rho^{(0)}=0, \tag{23}$$

$$\frac{\partial\delta}{\partial t}+\frac{1}{a}\frac{\partial v^i}{\partial x^i}+3\frac{\partial\Phi}{\partial t}=0, \tag{24}$$

$$\frac{\partial v^i}{\partial t}+Hv^i+\frac{1}{a}\frac{\partial\Psi}{\partial x^i}=0. \tag{25}$$

[8p]

Solution: The zero'th order equation just states that the total energy of CDM goes like $\rho^{(0)}=mn^{(0)}\propto 1/a^3\sim 1/V$ which makes sense since the number of CDM particles is constant, so the number density will just dilute with the expansion of the universe. The equation for δ states that the overdensity goes down if the divergence of the velocity field is positive (particles are on net moving away from the current point) or when space is locally stretching (since the volume increases but no new particles are created). The equation for v involves one term that takes into account the expansion of the universe (it wants to make $v\sim 1/a$), the second term is just the plain old gravitational force (gradient of the gravitaional potential).

e) Figure 1 shows the angular power spectrum for temperature fluctuations in the CMB.

- How would this curve change (roughly) if you changed the ratio of CDM to baryons (i.e. replacing some of the CDM by baryons or vise versa, keeping everything else fixed)? Please draw a sketch and explain your reasoning. [6p]

Solution: Increasing the amount of baryons would increase the baryon loading effect, leading to an enhancement of the odd numbered peaks in the power spectrum. This is because these peaks are compression peaks where the baryon and CDM overdensities are in phase (making the potential wells as large as possible), while for decompression peaks the baryons counteract the CDM overdensities. Increasing the amount of CDM (and removing baryons) would decrease this effect. Increasing the amount of baryons will also decrease the sound speed (and thus the sound horizon) of the photon-baryon fluid meaning that all acoustic peaks will shift to highet values of l. See fig. 2

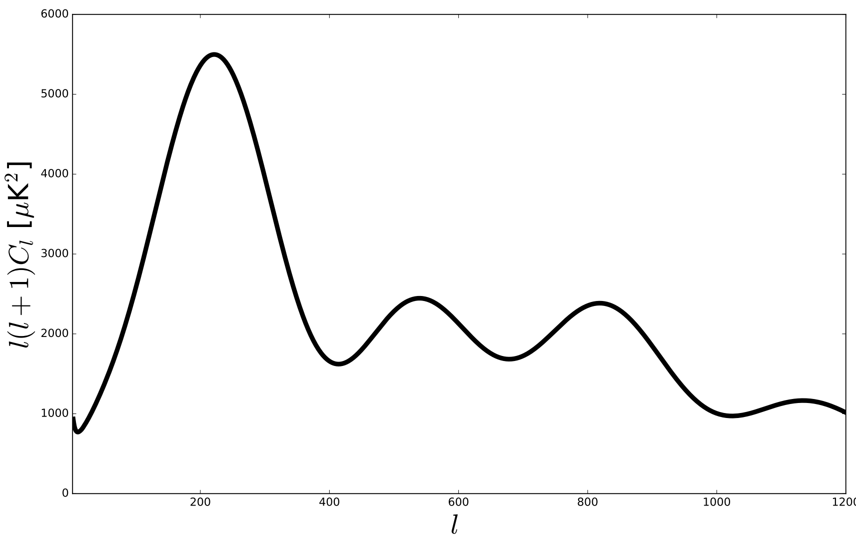

Figure 1: Angular power spectrum of temperature fluctuations in the CMB.

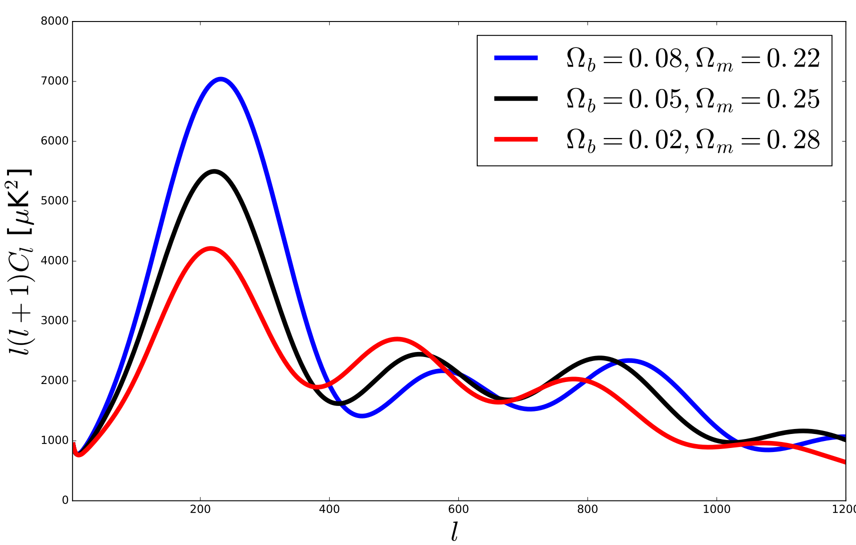

Figure 2: Angular power spectrum of temperature fluctuations in the CMB.

Problem 5 – Line-of-sight integration (AST9420) [30 p]

In this problem, we will derive the expression for the transfer function, $\Theta_l(k)$, used in for line-of-sight integration. Before we begin, let us review some relations concerning the Legendre polynomials, $P_l(\mu)$, that you may or may not find useful in the following:

$$
\begin{aligned}
P_0(\mu) &= 1 \\
P_1(\mu) &= \mu \\
P_l(\mu) &= (-1)^l P_l(-\mu) \\
\int_{-1}^{1} P_l(\mu)P_{l'}(\mu)d\mu &= \delta_{ll'}\frac{2}{2l+1} \\
j_l(x) &= \frac{(-i)^l}{2}\int_{-1}^{1} e^{i\mu x}P_l(\mu)d\mu \\
f_l &= \frac{i^l}{2}\int_{-1}^{1} f(\mu)P_l(\mu)d\mu
\end{aligned}
$$

Here $j_l(x)$ is the spherical Bessel function of order l, and $f(\mu)$ is an arbitrary function defined between -1 and 1.

Also, note that in the following, $\dot{}$ means derivative with respect to conformal time.

a) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

where $\Theta = \Theta(k, \mu, \eta)$ and $\mu \equiv \hat{k}\cdot\hat{p}$. Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}[\Theta_0 + \mu v_b],$$

and show that this equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S}e^{ik\mu(\eta-\eta_0)-\tau}d\eta.$$

Hint: Note that

$$\dot{\Theta} + (ik\mu - \dot{\tau})\Theta = e^{-ik\mu\eta+\tau}\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right].$$

[6p]

Solution: Starting with:

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

we can rewrite it as

$$
\begin{aligned}
\dot{\Theta} + (ik\mu - \dot{\tau})\Theta &= -\dot{\Phi} - ik\mu\Psi - \dot{\tau}[\Theta_0 + \mu v_b] \\
&= \tilde{S}.
\end{aligned}
$$

or simply

$$e^{-ik\mu\eta+\tau}\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right]=\tilde{S}.$$

multiplying with the exponential, and integrating over η from some very early time $\eta_{\mathbf{init}}$ (where $\tau(\eta_{\mathbf{init}})\gg 1$) gives

$$\left[\Theta e^{ik\mu\eta-\tau}\right]_{\eta_{\mathbf{init}}}^{\eta_0}=\int_{\eta_{\mathbf{init}}}^{\eta_0}d\eta\tilde{S}e^{ik\mu\eta-\tau},$$

$$\Theta(\eta_0)\exp\left(ik\mu\eta_0-\underbrace{\tau(\eta_0)}_{=0}\right)+\underbrace{\Theta(\eta_{\mathbf{init}})\exp\left(ik\mu\eta_{\mathbf{init}}-\underbrace{\tau(\eta_{\mathbf{init}})}_{\gg 1}\right)}_{=0}=\int_{\eta_{\mathbf{init}}}^{\eta_0}d\eta\tilde{S}e^{ik\mu\eta-\tau}.$$

Multiplying by the exponential, and realizing that that the integral from $\eta=0$ to $\eta_{\mathbf{init}}$ is negligible (because $e^{-\tau(\eta)}\approx 0$ at these really early times), we get

$$\Theta(\eta_0,k,\mu)=\int_0^{\eta_0}d\eta\tilde{S}e^{ik\mu(\eta-\eta_0)-\tau}.$$

b) Assume that $\tilde{S}$ does not depend on μ (in this sub-problem only). Show that in this case

$$\Theta_l(\eta_0,k)=(-1)^l\int_0^{\eta_0}\tilde{S}e^{-\tau}j_l[k(\eta-\eta_0)]d\eta,$$

where $\Theta_l(\eta,k)$ are the multipole expansion coefficients of $\Theta(\eta,k,\mu)$. [4p]

Solution:

$$\begin{aligned}\Theta_l(\eta_0,k)=\frac{i^l}{2}\int_{-1}^{1}d\mu\Theta(\mu)P_l(\mu)&=\frac{i^l}{2}\int_{-1}^{1}d\mu\int_0^{\eta_0}d\eta\tilde{S}e^{ik\mu(\eta-\eta_0)-\tau}P_l(\mu)\\&=\int_0^{\eta_0}d\eta\tilde{S}e^{-\tau}\frac{i^l}{2}\int_{-1}^{1}d\mu e^{ik\mu(\eta-\eta_0)}P_l(\mu)\\&=\int_0^{\eta_0}d\eta\tilde{S}e^{-\tau}(-1)^l\frac{(-i)^l}{2}\int_{-1}^{1}d\mu e^{ik\mu(\eta-\eta_0)}P_l(\mu)\\&=(-1)^l\int_0^{\eta_0}d\eta\tilde{S}e^{-\tau}j_l(k[\eta-\eta_0]).\end{aligned}$$

c) In reality, $\tilde{S}$ does of course depend on μ, and this has to be taken into account. The easiest way of doing this is by noting that $\tilde{S}$ is multiplied with $e^{ik\mu(\eta-\eta_0)}$, and μ and $k(\eta-\eta_0)$ are therefore Fourier conjugate (just like k and x). This allows us to set

$$\mu\rightarrow\frac{1}{ik}\frac{d}{d\eta}$$

everywhere μ appears in $\tilde{S}$, just like we can set $ik\rightarrow d/dx$ in a standard Fourier transformation.

Use this to show that the full solution for the transfer function is

$$\Theta_l(\eta_0, k) = \int_0^{\eta_0} S(k,\eta) j_l[k(\eta_0 - \eta)] d\eta,$$

where

$$S(k,\eta) = e^{-\tau}\left[-\dot{\Phi} - \dot{\tau}\Theta_0\right] + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{iv_b\dot{\tau}}{k}\right)\right] \tag{26}$$

Hint: You may need your old knowledge about integration-by-parts to get this right. [8p]

Solution:

$$\begin{aligned}
\Theta_l(\eta_0, k) &= \frac{i^l}{2}\int_{-1}^{1} d\mu \int_0^{\eta_0} d\eta \left[-\dot{\Phi} - \dot{\tau}\Theta_0 - ik\mu\left(\Psi - \frac{iv_b\dot{\tau}}{k}\right)\right] e^{ik\mu(\eta-\eta_0)-\tau} P_l(\mu) \\
&= \frac{i^l}{2}\int_{-1}^{1} d\mu \int_0^{\eta_0} d\eta \left[-(\dot{\Phi} + \dot{\tau}\Theta_0)e^{ik\mu(\eta_0-\eta)} - \frac{d}{d\eta}\left(e^{ik\mu(\eta-\eta_0)}\right)\left(\Psi - \frac{iv_b\dot{\tau}}{k}\right)\right] e^{-\tau} P_l(\mu) \\
&= \frac{i^l}{2}\int_{-1}^{1} d\mu \int_0^{\eta_0} d\eta \left[-e^{-\tau}(\dot{\Phi} + \dot{\tau}\Theta_0) + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{iv_b\dot{\tau}}{k}\right)\right]\right] e^{ik\mu(\eta-\eta_0)} P_l(\mu) \\
&= \frac{(-i)^l}{2}\int_{-1}^{1} d\mu \int_0^{\eta_0} d\eta \left[-e^{-\tau}(\dot{\Phi} + \dot{\tau}\Theta_0) + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{iv_b\dot{\tau}}{k}\right)\right]\right] e^{ik\mu(\eta-\eta_0)} P_l(-\mu) \\
&= \frac{(-i)^l}{2}\int_{-1}^{1} d\mu \int_0^{\eta_0} d\eta \left[-e^{-\tau}(\dot{\Phi} + \dot{\tau}\Theta_0) + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{iv_b\dot{\tau}}{k}\right)\right]\right] e^{ik\mu(\eta_0-\eta)} P_l(\mu) \\
&= \int_0^{\eta_0} d\eta \left[-e^{-\tau}(\dot{\Phi} + \dot{\tau}\Theta_0) + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{iv_b\dot{\tau}}{k}\right)\right]\right] j_l[k(\eta_0-\eta)] \\
&= \int_0^{\eta_0} d\eta S(k,\eta) j_l[k(\eta_0-\eta)].
\end{aligned}$$

d) Eq. 26 can be rewritten in the following form

$$S(k,\eta) = g(\eta)\left[\Theta_0 + \Psi\right] + \frac{d}{d\eta}\left(\frac{iv_b g(\eta)}{k}\right) + e^{-\tau}\left[\dot{\Psi} - \dot{\Phi}\right]. \tag{27}$$

- What is $g(\eta)$ here? (Explain physically!)

Solution: $g(\eta) = -\dot{\tau}e^{-\tau}$ is the visibility function, it denotes the probability density for a given photon to scatter for the last time at a time η.

- Explain what each of these terms means physically (You can draw a sketch if you want to). [12p]

Solution: The first term, called the Sachs-Wolfe term, is the most important term. Since it is multiplied by $g(\eta)$, it is evaluated at last scattering. It shows that the temperature fluctuations we obeserve today arise from the temperature fluctuations at last scattering (the θ_0-term) after we take into account the blue/red-shift arising from the photons emerging a point with a high or low gravitaional potential (the Ψ-term). The second term

(containing v_b) is also evaluated at last scattering and corresponds to the doppler-shift arising from the fact that the photons are emitted from a moving baryon fluid. The third term is an integrated effect, meaning that it is evaluated along the line of sight of the photons moving from the last scattering surface to us, and it corresponds to the change in photon energy as it moves through a gravitational potential that is changing in time.

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{28}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{29}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{30}$$

where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

- For a perfect fluid, the energy-momentum tensor (in the rest frame of the fluid) is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{31}$$

where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{32}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{33}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{34}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{35}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{36}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{37}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{38}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{39}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{40}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{41}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{42}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{43}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{44}$$

1.4 Initial conditions

$$\Phi = 1 \tag{45}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{46}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{47}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{48}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{49}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{50}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{51}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{52}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{53}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{54}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{55}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{56}$$

- The Saha equation,

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{57}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b}\left[\beta(T_b)(1-X_e) - n_H \alpha^{(2)}(T_b) X_e^2\right], \tag{58}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{59}$$

$$\Lambda_{2s\to 1s} = 8.227\mathrm{s}^{-1} \tag{60}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{61}$$

$$n_{1s} = (1-X_e) n_H \tag{62}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \tag{63}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{64}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{65}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b) \tag{66}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \quad (67)$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \quad (68)$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \quad (69)$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \quad (70)$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \quad (71)$$