UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam for AST5220/9420 — Cosmology II

Date: Thursday, June 16th, 2016

Time: 09.00 – 13.00

The exam set consists of 9 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Note that the exam set this year is identical for AST5220 and AST9420. Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions

Answer each question with three or four sentences.

a) What does the equivalent principle in General Relativity state?

The equivalence principle states that an observer in free fall does not feel any forces, and can claim to be at rest. In this system, all laws of physics are locally equivalent to those of special relativity.

b) How does the energy density of 1) pressureless matter; 2) radiation; and 3) the cosmological constant scale with time? In each case, explain why using simply physical intuition.

1) The density of pressureless matter scales as a^{-3}, following the expansion of 3-dimensional space. 2) Radiation scales as a^{-4}, because in addition to expansion of space, the photon wavelength is also stretched proportionally to a. 3) The density of the cosmological constant is independent of a, as it represents the density of vacuum, which is independent of a.

c) The metric in the conformal Newtonian gauge reads

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)(dx^2 + dy^2 + dz^2). \tag{1}$$

What do Ψ and Φ represent physically in this expression, and what parameters do they depend on?

Ψ represents the Newtonian gravitational potential, and measures the "contraction" of time. Φ represents the local curvature potential, and measures the "contraction" of space. Both depend only on space and time, x^i and t.

d) Given a particle distribution function, $f(x^i, p^i, t)$, for a given particle species, define the mean density, $n(x^i, t)$, and velocity, $v(x^i, t)$. What does the distribution function measure?

The mean density and velocity read

$$n(x^i, t) = \int d^3p f(x^i, p^i, t)$$
$$v(x^i, t) = \frac{1}{n(x^i, t)} \int d^3p p^i f(x^i, p^i, t)$$

The distribution function measures the phase space particle density, ie., the number of particles per time-space-momentum volume element.

e) How many independent equations do Einstein's field equations correspond to in the conformal Newtonian gauge, and how many do they correspond to in the most general metric?

There are a total of 10 independent Einstein equations in the general case, corresponding to each of the elements in a symmetric 4×4 matrix. In the

conformal Newtonian gauge, these reduce to 2 because of spatial isotropy and homogeneity.

f) When deriving initial conditions for Φ in the conformal Newtonian gauge, one finds that the equations support two fundamentally different modes with respect to time, $\Phi(t)$. Why do we only care about one of these, and how does it depend on time early on?

The two solutions scale as $\mathcal{O}(a^{-3})$ and $\mathcal{O}(1)$, respectively. The first of these decay extremely rapidly, and only the second, and constant, solution survives to later time.

Problem 2 – Optical depth and recombination

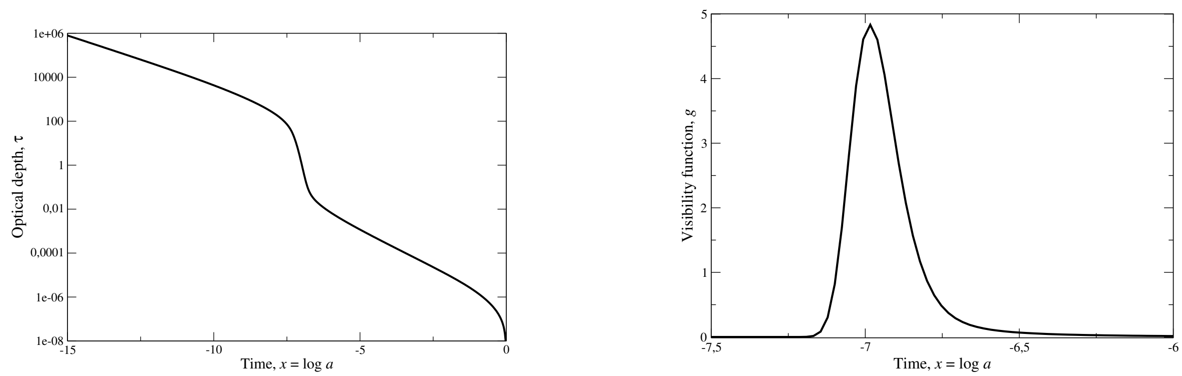

Figure 1: *Left:* The optical depth, τ, as a function of time. *Right:* The visibility function, g, as a function of time.

a) The left panel shows the optical depth as a function of cosmic time, as measured by $x = \log a$. Explain physically the behaviour of this curve. What is the numerical value of τ today, and why?

The optical depth primarily depends on the density of free electrons. This may be described by two functions of time, namely the baryon density, $n_b(t)$, and the free electron fraction, $X_e(t)$. Early in the evolution of the universe, all electrons are free, and $X_e = 1$. However, the optical depth still falls with time, because the baryon density falls as a^{-3}. Then something important happens around $x = -7$. At this time, the temperature of the universe drops below 3000K, corresponding roughly to the binding energy of hydrogen. Electrons and protons comine into neutral hydrogen, and X_e drops quickly. This is called recombination, and defines the time when the CMB photons are released from the cosmic fluid. This recombination does not last forever, though: After $x \approx -6$, the distance between remaining electrons becomes so large that it is difficult for an electron to find a partner, and the recombination rate slows down, following the general expansion rate of the universe again. At the very end, τ has to go to zero today, and so drops quickly.

b) What happens to this curve if you *increase* Ω_b? Explain your reasoning.

If you increase Ω_b, there will be more electrons available for photons to scatter on, ie., a higher n_b. This essentially corresponds to shifting the early-time part of the curve *up*. After recombination, though, things will be essentially as before, since all those extra electrons will have been recombined into neutral hydrogen.

c) What happens to this curve if you *increase* the CMB temperature, T_0? Explain your reasoning.

Increasing the temperature means that the universe will hit 3000K at a later time. As a result, recombination also happens later: The curve is essentially shifted to the *right*.

d) The right panel shows the visibility function, g. What is the physical interpretation of this function?

The visibility function is a probability distribution describing the probability for when a photon *scattered for the last time*.

e) If we had account for cosmic *reionization* in addition to recombination, releasing free electrons into the cosmic fluid when the first stars ignite, some 400 million years after the Big Bang, what would happen to the visibility function?

With free electrons emerging at a later time, X_e would increase. This would in turn create a low-level flat non-zero plateau at late times in the visibility function.

Problem 3 – The Saha equation

The Saha equation plays a central role when calculating the CMB power spectrum. In the following, we will derive one form of this equation suitable for this purpose.

For a gas consisting of photons, protons and electrons in thermodynamic equilibrium, one can show that the following relation holds

$$\frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} = \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}},$$

where n_X is the density of species X, and

$$n_X^{(0)} = \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\frac{E_X}{kT}}$$

is the equilibrium density. (Here, p denotes momentum of the particle, $E_X = \sqrt{p_X^2 c^2 + m_X^2 c^4}$ is the energy, and T is the common equilibrium temperature.) We will only consider cases for which $mc^2 >> kT$, ie., systems for which the temperature is much lower than the rest mass of the particles.

Next, define the free electron fraction to be

$$X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}, \tag{2}$$

where the latter equality holds due to the requirement of a neutral universe.

Finally, you can assume as known that the photon density equals the equilibrium density during thermodynamic equilibrium, $n_\gamma = n_\gamma^{(0)}$.

a) Show that the Saha equation may be written on the form

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

See lecture notes.

b) Show that the background density of (massive) species X is given by

$$n_X^{(0)} = \left(\frac{kTm_X}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{m_X c^2}{kT}}.$$

Hint: You may need to know that

$$\int_0^\infty \sqrt{u} e^{-u} du = \frac{\sqrt{\pi}}{2}.$$

See lecture notes.

c) Finally, show that the full Saha equation for the electron density is given by

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \left(\frac{kTm_e}{2\pi\hbar^2} \right)^{3/2} e^{-\frac{\epsilon_0}{kT}}.$$

What is ϵ_0 here? Which assumption regarding m_p and m_H is used here?

See lecture notes. $epsilon_0$ **is the binding energy of hydrogen. We assume that** $m_p \approx m_H$**.**

d) Why can't we use the Saha equation at all times, but must instead use the Peebles equation at late times?

The Saha equation only holds during strong thermodynamic equilibrium, when the atomic interaction rates are much faster than the expansion of the universe. Once recombination sets in, this no longer holds, as the distance between free electrons become large. At this time, the full Peebles equation must be used, accounting both for quantum physics and universal expansion.

Problem 4 – The power spectrum of tensor fluctuations

In this problem we will revisit the power spectrum of tensor fluctuations, which plays an important part in defining the initial conditions for the Einstein-Boltzmann equations. First, recall that the metric for tensor perturbations travelling along the z-axis is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2(1+h_+) & a^2 h_x & 0 \\ 0 & a^2 h_x & a^2(1-h_+) & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}, \tag{3}$$

where h_+ and h_x are small perturbation amplitudes.

a) Outline schematically the mathematical/operational steps that are needed in order to derive Einstein's equations from this metric, which reads

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} + k^2 h = 0, \tag{4}$$

where $h = \{h_x, h_+\}$. (But don't actually perform the steps! :-))

(a) Derive all Christoffel symbols, $\Gamma^{\mu}_{\alpha\beta}$.
(b) Derive Ricci tensor, $R_{\mu\nu}$.
(c) Derive Ricci scalar, $R = R^{\mu}_{\nu}$.
(d) Derive Einstein equations, $E_{\mu\nu} = T_{\mu\nu}$.

b) What sort of an equation is this? What is the effect of the second term?

This is a damped harmonic oscillator. The second term is an effective friction term due to Hubble expansion.

c) Roughly sketch the solution of this equation as a function of time for three different values of k.

d) After quantizing this the above solution, one arrives at the following expression for the power spectrum of tensor fluctuations,

$$P_h(k) = \frac{8\pi G H^2}{2k^3}, \tag{5}$$

where H is the Hubble expansion factor during inflation. What does this expression describe physically, and how does it relate to solving the Einstein-Boltzmann equations?

This expression measures the power spectrum of initial tensor perturbations, created during inflation, ie., the squared-amplitude of Fourier modes as a function of wavelength. The higher the power spectrum, the stronger the tensor fluctuations. According to the standard picture, these perturbations created corresponding perturbations in the curvature potential, Φ, and thereby also set up a corresponding perturbation spectrum in each of the other perturbation quantities, such as δ, δ_b, Θ_0 etc. This spectrum thus defines the *initial conditions* for the Einstein-Boltzmann equations.

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{6}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{7}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{8}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.
- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{9}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{10}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{11}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{12}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{13}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{14}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{15}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{16}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{17}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{18}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{19}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{20}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{21}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{22}$$

1.4 Initial conditions

$$\Phi = 1 \tag{23}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{24}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{25}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{26}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{27}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{28}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{29}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{30}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{31}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{32}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{33}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{34}$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b}, \tag{35}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \tag{36}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{37}$$

$$\Lambda_{2s\to 1s} = 8.227\mathrm{s}^{-1} \tag{38}$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{39}$$

$$n_{1s} = (1 - X_e) n_H \tag{40}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \tag{41}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b} \tag{42}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \tag{43}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0/T_b) \tag{44}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{45}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{46}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{47}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{48}$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{49}$$