UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220/9420 — Cosmology II

Date: Thursday, June 16th, 2016

Time: 09.00 – 13.00

The exam set consists of 9 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Note that the exam set this year is identical for AST5220 and AST9420. Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions

Answer each question with three or four sentences.

a) What does the equivalent principle in General Relativity state?

b) How does the energy density of 1) pressureless matter; 2) radiation; and 3) the cosmological constant scale with time? In each case, explain why using simply physical intuition.

c) The metric in the conformal Newtonian gauge reads

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)(dx^2 + dy^2 + dz^2). \tag{1}$$

What do Ψ and Φ represent physically in this expression, and what parameters do they depend on?

d) Given a particle distribution function, $f(x^i, p^i, t)$, for a given particle species, define the mean density, $n(x^i, t)$, and velocity, $v(x^i, t)$. What does the distribution function measure?

e) How many independent equations do Einstein's field equations correspond to in the conformal Newtonian gauge, and how many do they correspond to in the most general metric?

f) When deriving initial conditions for Φ in the conformal Newtonian gauge, one finds that the equations support two fundamentally different modes with respect to time, $\Phi(t)$. Why do we only care about one of these, and how does it depend on time early on?

Problem 2 – Optical depth and recombination

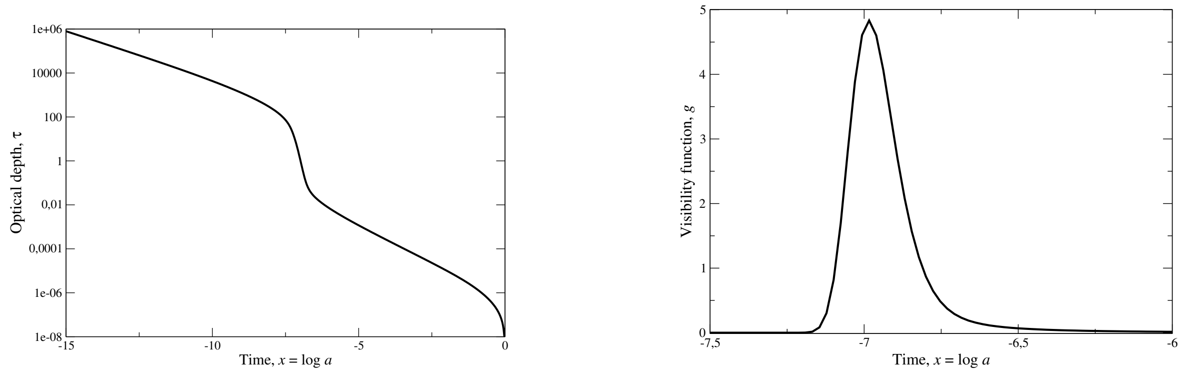

Figure 1: *Left:* The optical depth, τ, as a function of time. *Right:* The visibility function, g, as a function of time.

a) The left panel shows the optical depth as a function of cosmic time, as measured by $x = \log a$. Explain physically the behaviour of this curve. What is the numerical value of τ today, and why?

b) What happens to this curve if you *increase* Ω_b? Explain your reasoning.

c) What happens to this curve if you *increase* the CMB temperature, T_0? Explain your reasoning.

d) The right panel shows the visibility function, g. What is the physical interpretation of this function?

e) If we had account for cosmic *reionization* in addition to recombination, releasing free electrons into the cosmic fluid when the first stars ignite, some 400 million years after the Big Bang, what would happen to the visibility function?

Problem 3 – The Saha equation

The Saha equation plays a central role when calculating the CMB power spectrum. In the following, we will derive one form of this equation suitable for this purpose.

For a gas consisting of photons, protons and electrons in thermodynamic equilibrium, one can show that the following relation holds

$$\frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} = \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}},$$

where n_X is the density of species X, and

$$n_X^{(0)} = \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\frac{E_X}{kT}}$$

is the equilibrium density. (Here, p denotes momentum of the particle, $E_X = \sqrt{p_X^2 c^2 + m_X^2 c^4}$ is the energy, and T is the common equilibrium temperature.) We will only consider cases for which $mc^2 >> kT$, ie., systems for which the temperature is much lower than the rest mass of the particles.

Next, define the free electron fraction to be

$$X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}, \tag{2}$$

where the latter equality holds due to the requirement of a neutral universe.

Finally, you can assume as known that the photon density equals the equilibrium density during thermodynamic equilibrium, $n_\gamma = n_\gamma^{(0)}$.

a) Show that the Saha equation may be written on the form

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

b) Show that the background density of (massive) species X is given by

$$n_X^{(0)} = \left(\frac{kTm_X}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{m_X c^2}{kT}}.$$

Hint: You may need to know that

$$\int_0^\infty \sqrt{u} e^{-u} du = \frac{\sqrt{\pi}}{2}.$$

c) Finally, show that the full Saha equation for the electron density is given by

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \left(\frac{kTm_e}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{\epsilon_0}{kT}}.$$

What is ϵ_0 here? Which assumption regarding m_p and m_H is used here?

d) Why can't we use the Saha equation at all times, but must instead use the Peebles equation at late times?

Problem 4 – The power spectrum of tensor fluctuations

In this problem we will revisit the power spectrum of tensor fluctuations, which plays an important part in defining the initial conditions for the Einstein-Boltzmann equations. First, recall that the metric for tensor perturbations travelling along the z-axis is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2(1+h_+) & a^2 h_x & 0 \\ 0 & a^2 h_x & a^2(1-h_+) & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}, \tag{3}$$

where h_+ and h_x are small perturbation amplitudes.

a) Outline schematically the mathematical/operational steps that are needed in order to derive Einstein's equations from this metric, which reads

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} + k^2 = 0, \tag{4}$$

where $h = \{h_x, h_+\}$. (But don't actually perform the steps! :-))

b) What sort of an equation is this? What is the effect of the second term?

c) Roughly sketch the solution of this equation as a function of time for three different values of k.

d) After quantizing this the above solution, one arrives at the following expression for the power spectrum of tensor fluctuations,

$$P_h(k) = \frac{8\pi G H^2}{2k^3}, \tag{5}$$

where H is the Hubble expansion factor during inflation. What does this expression describe physically, and how does it relate to solving the Einstein-Boltzmann equations?

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{6}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{7}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{8}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.
- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{9}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{10}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{11}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{12}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{13}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{14}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{15}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{16}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{17}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{18}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{19}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{20}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{21}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{22}$$

1.4 Initial conditions

$$\Phi = 1 \tag{23}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{24}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{25}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{26}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{27}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{28}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{29}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{30}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{31}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{32}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{33}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{34}$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b}, \tag{35}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \tag{36}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to1s} + \Lambda_\alpha}{\Lambda_{2s\to1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{37}$$

$$\Lambda_{2s\to1s} = 8.227\mathrm{s}^{-1} \tag{38}$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{39}$$

$$n_{1s} = (1 - X_e) n_H \tag{40}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \tag{41}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b} \tag{42}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \tag{43}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0/T_b) \tag{44}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{45}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{46}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{47}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{48}$$

3. The CMB spectrum:

$$C_l = \int_0^{\infty} \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{49}$$