UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220/9420 — Cosmology II

Date: Thursday, June 11th, 2015

Time: 09.00 – 13.00

The exam set consists of 11 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Note that AST5220 students are supposed to answer problems 1)-4), while AST9420 students answer problems 1)-3) and 5). Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions (AST5220 and AST9420)

Answer each question with three or four sentences.

a) Write down the Boltzmann equation on schematic form for a general distribution function f. What is the physical interpretation of f?

Answer:

$$\frac{df}{dt} = C[f] \tag{1}$$

$f = f(\vec{x}, t, \vec{p})$ **denotes the density of particles at a given position, $\vec{x}$, at a given time, t, moving with a given speed/momentum, $\vec{p}$.**

b) Why do we solve the Einstein-Boltzmann equations in Fourier- space instead of real-space?

Because it is much easier to solve N independent ordinary differential equations than it is to solve a set of strongly coupled partial differential equations.

c) Why is it acceptable to set the curvature potential, Φ, to 1 at early times when solving the Einstein-Boltzmann equations?

Because the equations are linear. The exact numerical value of Φ may therefore be inserted at the end, after completing all calculations.

d) The Boltzmann equation for two-particle processes reads

$$a^{-3}\frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right), \tag{2}$$

where n_i is the density of particle species $i = \{1, 2, 3, 4\}$, $n_i^{(0)}$ is the corresponding average density, $\langle \sigma v \rangle$ is a thermally averaged cross-section, and a is the usual cosmological scale factor. What is the corresponding Saha equation, and why does it hold?

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \tag{3}$$

This equation holds because the typical time scale for the expansion rate of the universe on the left-hand side of the equation is much smaller than the typical time scale for the particle interactions on the right-hand side. The only way to make this equation hold, is therefore to make the last factor very close to zero.

e) The Einstein equation reads

$$E_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{4}$$

What does the left- and right-hand sides of this equation describe, respectively?

The left-hand side of the equation is the Einstein tensor, which depends only on the metric, and therefore describes geometry of space. The right-hand side is the energy-momentum tensor, and describes the energy-matter content of space.

f) What is the main advantage of the line-of-sight integration method for computing the CMB power spectrum?

The main advantage is computational speed. By formally integrating the Boltzmann equations before expanding into multipole moments, as opposed to expanding before integrating, only only has to solve for ≈ 6 photon multipole moments, rather than $\ell_{\mathbf{max}}$ moments, even when calculating the full CMB spectrum to high ℓ's. A second advantage is a clearer physical interpretation of the various effects that impact the spectrum, such as Sachs-Wolfe, Doppler, Integrated Sachs-Wolfe etc.

Problem 2 – Cosmological parameters (AST5220 and AST9420)

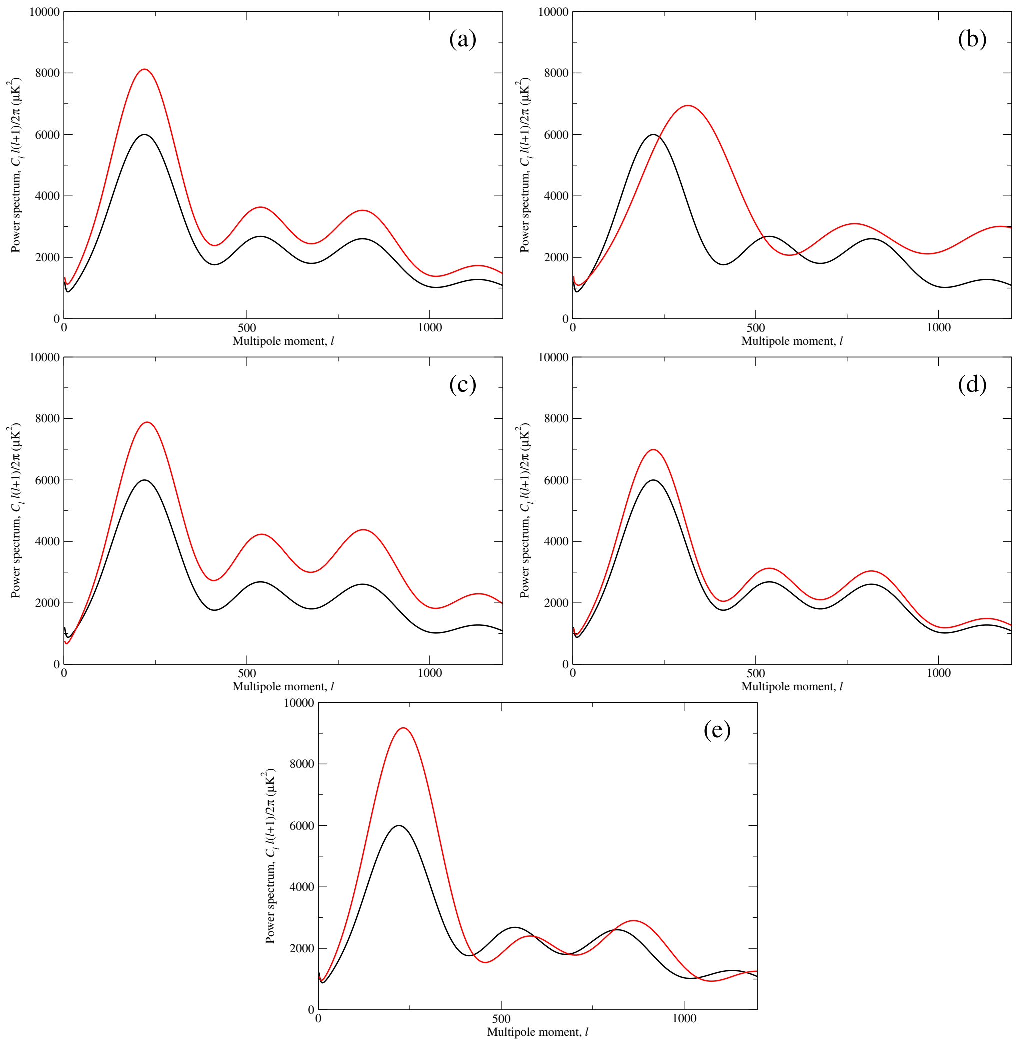

Figure 1: CMB spectra for different cosmological parameters.

Each of the panels marked (a)-(e) shows the CMB temperature power spectrum for two different cosmological parameter combinations. The black curve shows the standard best-fit ΛCDM spectrum, while the red curve shows a spectrum for which *one* parameter has been changed. **In each case, state which parameter was changed, and describe why it must be that parameter.**

Answer:

a) A_s – ratio between the red and black curves is constant, corresponding to a direct scaling.

b) Ω_k, Ω_{Lambda} or Ω_0 – the peaks are significantly shifted to higher ℓ's. It cannot be Ω_b or Ω_m since the relative height of the peaks is essentially unchanged.

c) n_s – the red spectrum is *lower* than the black spectrum at very low ℓ's,

but higher at high ℓ's.

d) τ – optical depth of reionization is lower in the red curve. It looks very much like an n_s effect, but the red spectrum crosses at lower ℓ's, and just barely. However, this shows why τ and n_s are strongly degenerate in the temperature power spectrum.

e) Ω_b – the first and third peak are higher relative to the second peak.

Problem 3 – The geodesic equation (AST5220 and AST9420)

In this problem we will consider the geodesic equation.

a) What does Einstein's equivalence principle state regarding an observer in free fall?

Answer: An observer in free fall can claim to be a rest. In his/her reference frame all laws of physics are locally identical to those of special relativity.

b) The equation of motion for a particle in free fall reads

$$\frac{d^2\xi^\mu}{d\tau^2} = 0, \tag{5}$$

where ξ^μ is the four-position in a coordinate system following the particle, and τ is the eigentime. Derive the geodesic equation from this by performing a coordinate transformation to an arbitrary coordinate system x^μ. It may be useful to know that the Christoffel symbols may be written as

$$\Gamma^\mu_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\rho}\frac{\partial^2 \xi^\rho}{\partial x^\alpha \partial x^\beta} \tag{6}$$

Answer: See lecture notes.

c) What are the numerical values of the Christoffel symbols in Euclidean space? What does this tell us about a free particle moving in a flat space?

Answer: Zero. This tells us that a free particle moving in a flat space moves along a straight line, as per Newton's second law.

d) The synchronous gauge is defined through the following line element,

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j \tag{7}$$

where

$$h_{ij} = \begin{bmatrix} -2\phi & 0 & 0 \\ 0 & -2\phi & 0 \\ 0 & 0 & h+4\phi \end{bmatrix}, \tag{8}$$

and $\phi = \phi(x,t)$, $h = h(x,t)$ are two general perturbation fields. Consider the special case with $\phi = 0$, and compute Γ^0_{11} and Γ^3_{33}. (PS! The general expression for the Christoffel symbols in terms of the metric is given in the Appendix.)

Answer:

$$\Gamma^0_{11} = Ha^2 \tag{9}$$

$$\Gamma^3_{33} = \frac{1}{2}\frac{\partial h}{\partial z} \tag{10}$$

$$\tag{11}$$

Problem 4 – The Boltzmann equation for free photons (AST5220)

In this problem you will derive a few critical components needed for the Boltzmann equation for free photons, ie., neglecting any collision terms, in the conformal Newtonian gauge,

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)(dx^2+dy^2+dz^2).$$

a) For a massless photon one has $g_{\mu\nu}P^\mu P^\nu = 0$, where $P^\mu = dx^\mu/d\lambda$ is the four-momentum of the photon. Define $p^2 = g_{ij}P^iP^j$. Show that

$$\begin{aligned} P^0 &= p(1-\Psi) \\ P^i &= \frac{p}{a}\hat{p}(1-\Phi) \end{aligned}$$

to first order in Ψ and Φ, where $\hat{p}$ is the unit vector pointing along the 3-momentum of the photon.

Answer: See lecture notes.

b) Show that

$$\frac{dx^i}{dt} = \frac{\hat{p}}{a}(1-\Phi+\Psi)$$

to first order. Give a physical interpretation of this equation.

Answer: See lecture notes. This equation describes the motion of a photon through space in co-moving units. The motion is slower when a is larger, since the universe is then effectively bigger; it is also slower when moving through an over-density, both because time slows slightly down ($\psi < 0$), and because space is slightly stretched ($\phi > 0$).

c) Write down the Boltzmann equation schematically in term of partial derivatives over each of its dynamic variables, x^i, t, p and $\hat{p}^i$. Why can the term dependent on $\hat{p}$ be neglected?

Answer:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt} + \frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt} = 0 \qquad (12)$$

The last term can be neglected because it must be at least second order in the perturbation variables; the Maxwell distribution does not depend on $\hat{p}$ to zeroth order, and the direction of motion does not change by itself in a uniform universe.

d) Starting from the 0-component of the geodesic equation, show that

$$\frac{dp}{dt} = p\frac{d\Psi}{dt} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi)$$

to first order.

Answer: See lecture notes.

e) Inserting both b) and d) into c), adopting a Bose-Einstein distribution to first order in Θ, adopting conformal time, and Fourier transforming the equation eventually yields the following expression:

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b], \tag{13}$$

where $\Theta = \Theta_k(\eta, \mu)$ etc. From this expression, derive the corresponding differential equation for the monopole expansion, Θ_0, knowing that the two lowest-order Legendre polynomials are $P_0(\mu) = 1$ and $P_1(\mu) = \mu$, and that $\Theta_l = \frac{i^l}{2}\int_{-1}^{1} \Theta(\mu)P_l(\mu)d\mu$.

Answer: See lecture notes for derivation. The equation reads

$$\dot{\Theta}_0 = -k\Theta_0 - \dot{\Phi} \tag{14}$$

Problem 5 – Line-of-sight integration (AST9420)

In this problem, we will derive the expression for the transfer function, $\Theta_l(k)$, used in for line-of-sight integration. Before we begin, let us review some relations concerning the Legendre polynomials, $P_l(\mu)$, that you may or may not find useful in the following:

$$\begin{aligned}
P_0(\mu) &= 1 \\
P_1(\mu) &= \mu \\
P_l(\mu) &= (-1)^l P_l(-\mu) \\
\int_{-1}^{1} P_l(\mu)P_{l'}(\mu)d\mu &= \delta_{ll'}\frac{2}{2l+1} \\
j_l(x) &= \frac{i^l}{2}\int_{-1}^{1} e^{-i\mu x}P_l(\mu)d\mu \\
f_l &= \frac{i^l}{2}\int_{-1}^{1} f(\mu)P_l(\mu)d\mu
\end{aligned}$$

Here $j_l(x)$ is the spherical Bessel function of order l, and $f(\mu)$ is an arbitrary function defined between -1 and 1.

Also, note that in the following, $\dot{}$ means derivative with respect to conformal time.

a) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

where $\Theta = \Theta(k, \mu, \eta)$ and μ is the angle between the photon propagation direction, $\hat{p}$, and the wave vector, $\hat{k}$. Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

and show that this equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S} e^{ik\mu(\eta-\eta_0)-\tau} d\eta.$$

(Note that we have dropped a quadrupole/polarization term in this expression, in order to keep things simple(r).)

Answer: See lecture notes.

b) Assume that $\tilde{S}$ does not depend on μ (in this sub-problem only). Show that in this case

$$\Theta_l(\eta_0, k) = (-1)^l \int_0^{\eta_0} \tilde{S} e^{-\tau} j_l[k(\eta_0 - \eta)]d\eta,$$

where $\Theta_l(\eta, k)$ are the multipole expansion coefficients of $\Theta(\eta, k, \mu)$.

Answer: See lecture notes.

c) In reality, $\tilde{S}$ does of course depend on μ, and this have to be taken into account in the expression in c). The easiest way of doing this is by noting that $\tilde{S}$ is multiplied with $e^{ik\mu(\eta-\eta_0)}$, and μ and $k(\eta-\eta_0)$ are therefore Fourier conjugate (just like k and x). This allows us to set

$$\mu \rightarrow \frac{1}{ik}\frac{d}{d\eta}$$

everywhere μ appears in $\tilde{S}$, just like we can set $ik \rightarrow d/dx$ in a standard Fourier transformation.

Use this to show that the full solution for the transfer function is

$$\Theta_l(\eta_0, k) = \int_0^{\eta_0} S(k,\eta) j_l[k(\eta_0 - \eta)] d\eta,$$

where

$$S(k,\eta) = e^{-\tau}\left[-\dot{\Phi} - \dot{\tau}\Theta_0\right] + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{v_b\dot{\tau}}{k}\right)\right]$$

(Hint: You may need your old knowledge about integration-by-parts to get this right :-))

Answer: See lecture notes.

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{15}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{16}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{17}$$

where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{18}$$

where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{19}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{20}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{21}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{22}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{23}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{24}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{25}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{26}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{27}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{28}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{29}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{30}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{31}$$

1.4 Initial conditions

$$\Phi = 1 \tag{32}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{33}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{34}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{35}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{36}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{37}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{38}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{39}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{40}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{41}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{42}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{43}$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b}, \tag{44}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \tag{45}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{46}$$

$$\Lambda_{2s\to 1s} = 8.227\mathrm{s}^{-1} \tag{47}$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{48}$$

$$n_{1s} = (1 - X_e) n_H \tag{49}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \tag{50}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b} \tag{51}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \tag{52}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0/T_b) \tag{53}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{54}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{55}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{56}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{57}$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{58}$$