

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam for AST5220/9420 — Cosmology II

Date: Thursday, June 11th, 2015

Time: 09.00 – 13.00

The exam set consists of 11 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Note that AST5220 students are supposed to answer problems 1)-4), while AST9420 students answer problems 1)-3) and 5). Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions (AST5220 and AST9420)

Answer each question with three or four sentences.

- a) Write down the Boltzmann equation on schematic form for a general distribution function f . What is the physical interpretation of f ?
- b) Why do we solve the Einstein-Boltzmann equations in Fourier- space instead of real-space?
- c) Why is it acceptable to set the curvature potential, Φ , to 1 at early times when solving the Einstein-Boltzmann equations?
- d) The Boltzmann equation for two-particle processes reads

$$a^{-3} \frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right), \quad (1)$$

where n_i is the density of particle species $i = \{1, 2, 3, 4\}$, $n_i^{(0)}$ is the corresponding average density, $\langle \sigma v \rangle$ is a thermally averaged cross-section, and a is the usual cosmological scale factor. What is the corresponding Saha equation, and why does it hold?

- e) The Einstein equation reads

$$E_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (2)$$

What does the left- and right-hand sides of this equation describe, respectively?

- f) What is the main advantage of the line-of-sight integration method for computing the CMB power spectrum?

Problem 2 – Cosmological parameters (AST5220 and AST9420)

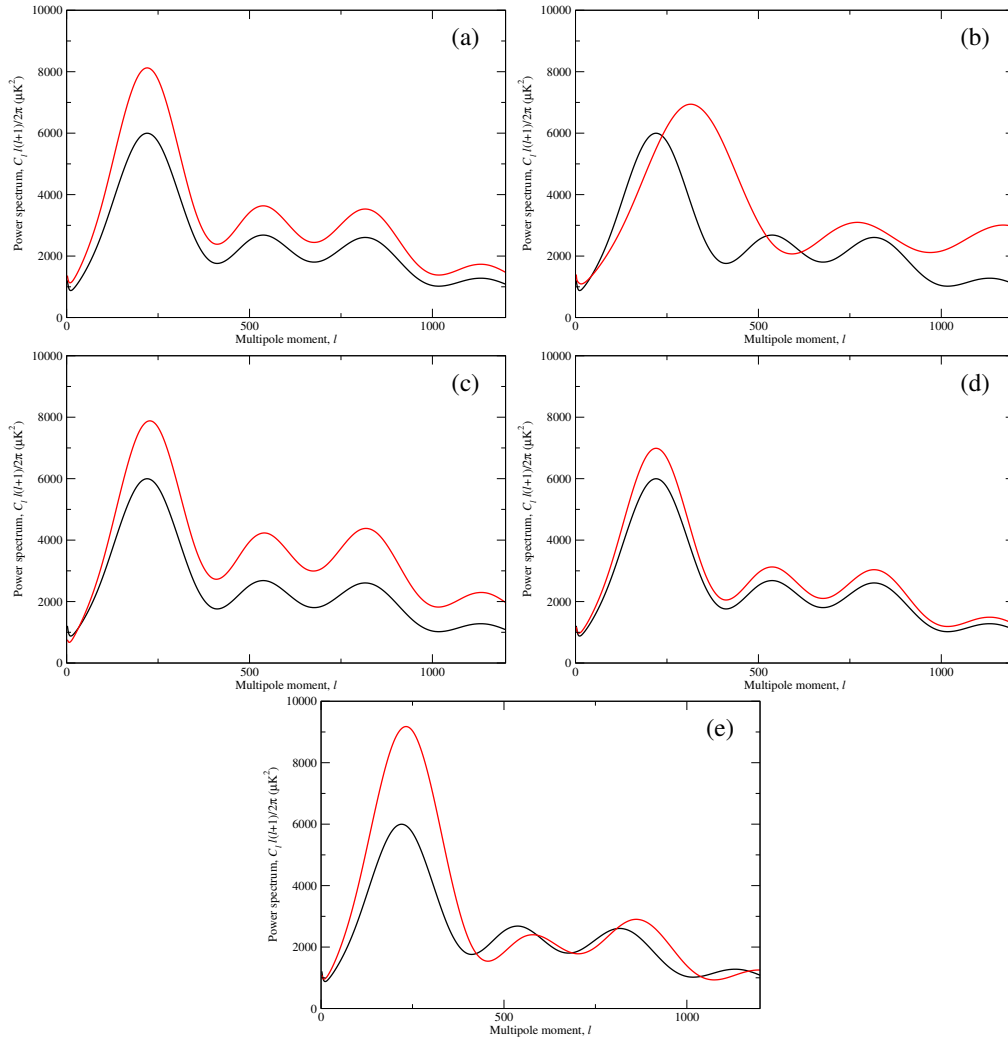


Figure 1: CMB spectra for different cosmological parameters.

Each of the panels marked (a)-(e) shows the CMB temperature power spectrum for two different cosmological parameter combinations. The black curve shows the standard best-fit Λ CDM spectrum, while the red curve shows a spectrum for which *one* parameter has been changed. **In each case, state which parameter was changed, and describe why it must be that parameter.**

Problem 3 – The geodesic equation (AST5220 and AST9420)

In this problem we will consider the geodesic equation.

- a) What does Einstein's equivalence principle state regarding an observer in free fall?
- b) The equation of motion for a particle in free fall reads

$$\frac{d^2\xi^\mu}{d\tau^2} = 0, \quad (3)$$

where ξ^μ is the four-position in a coordinate system following the particle, and τ is the eigentime. Derive the geodesic equation from this by performing a coordinate transformation to an arbitrary coordinate system x^μ . It may be useful to know that the Christoffel symbols may be written as

$$\Gamma_{\alpha\beta}^\mu = \frac{\partial x^\mu}{\partial \xi^\rho} \frac{\partial^2 \xi^\rho}{\partial x^\alpha \partial x^\beta} \quad (4)$$

- c) What are the numerical values of the Christoffel symbols in Euclidean space? What does this tell us about a free particle moving in a flat space?
- d) The synchronous gauge is defined through the following line element,

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j \quad (5)$$

where

$$h_{ij} = \begin{bmatrix} -2\phi & 0 & 0 \\ 0 & -2\phi & 0 \\ 0 & 0 & h + 4\phi \end{bmatrix}, \quad (6)$$

and $\phi = \phi(x, t)$, $h = h(x, t)$ are two general perturbation fields. Consider the special case with $\phi = 0$, and compute Γ_{11}^0 and Γ_{33}^3 . (PS! The general expression for the Christoffel symbols in terms of the metric is given in the Appendix.)

Problem 4 – The Boltzmann equation for free photons (AST5220)

In this problem you will derive a few critical components needed for the Boltzmann equation for free photons, ie., neglecting any collision terms, in the conformal Newtonian gauge,

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)(dx^2 + dy^2 + dz^2).$$

- a) For a massless photon one has $g_{\mu\nu}P^\mu P^\nu = 0$, where $P^\mu = dx^\mu/d\lambda$ is the four-momentum of the photon. Define $p^2 = g_{ij}P^i P^j$. Show that

$$\begin{aligned} P^0 &= p(1 - \Psi) \\ P^i &= \frac{p}{a}\hat{p}^i(1 - \Phi) \end{aligned}$$

to first order in Ψ and Φ , where \hat{p} is the unit vector pointing along the 3-momentum of the photon.

- b) Show that

$$\frac{dx^i}{dt} = \frac{\hat{p}^i}{a}(1 - \Phi + \Psi)$$

to first order. Give a physical interpretation of this equation.

- c) Write down the Boltzmann equation schematically in term of partial derivatives over each of its dynamic variables, x^i , t , p and \hat{p}^i . Why can the term dependent on \hat{p} be neglected?
- d) Starting from the 0-component of the geodesic equation, show that

$$\frac{dp}{dt} = p\frac{d\Psi}{dt} - \Gamma_{\alpha\beta}^0 \frac{P^\alpha P^\beta}{p}(1 + 2\Psi)$$

to first order.

- e) Inserting both b) and d) into c), adopting a Bose-Einstein distribution to first order in Θ , adopting conformal time, and Fourier transforming the equation eventually yields the following expression:

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b], \quad (7)$$

where $\Theta = \Theta_k(\eta, \mu)$ etc. From this expression, derive the corresponding differential equation for the monopole expansion, Θ_0 , knowing that the two lowest-order Legendre polynomials are $P_0(\mu) = 1$ and $P_1(\mu) = \mu$, and that $\Theta_l = \frac{i^l}{2} \int_{-1}^1 \Theta(\mu) P_l(\mu) d\mu$.

Problem 5 – Line-of-sight integration (AST9420)

In this problem, we will derive the expression for the transfer function, $\Theta_l(k)$, used in for line-of-sight integration. Before we begin, let us review some relations concerning the Legendre polynomials, $P_l(\mu)$, that you may or may not find useful in the following:

$$\begin{aligned}
 P_0(\mu) &= 1 \\
 P_1(\mu) &= \mu \\
 P_l(\mu) &= (-1)^l P_l(-\mu) \\
 \int_{-1}^1 P_l(\mu) P_{l'}(\mu) d\mu &= \delta_{ll'} \frac{2}{2l+1} \\
 j_l(x) &= \frac{i^l}{2} \int_{-1}^1 e^{-i\mu x} P_l(\mu) d\mu \\
 f_l &= \frac{i^l}{2} \int_{-1}^1 f(\mu) P_l(\mu) d\mu
 \end{aligned}$$

Here $j_l(x)$ is the spherical Bessel function of order l , and $f(\mu)$ is an arbitrary function defined between -1 and 1.

Also, note that in the following, $\dot{}$ means derivative with respect to conformal time.

- a) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

where $\Theta = \Theta(k, \mu, \eta)$ and μ is the angle between the photon propagation direction, \hat{p} , and the wave vector, \hat{k} . Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

and show that this equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S} e^{ik\mu(\eta-\eta_0)-\tau} d\eta.$$

(Note that we have dropped a quadrupole/polarization term in this expression, in order to keep things simple(r).)

- b) Assume that \tilde{S} does not depend on μ (in this sub-problem only). Show that in this case

$$\hat{\Theta}_l(\eta_0, k) = (-1)^l \int_0^{\eta_0} \tilde{S} e^{-\tau} j_l[k(\eta_0 - \eta)] d\eta,$$

where $\Theta_l(\eta, k)$ are the multipole expansion coefficients of $\Theta(\eta, k, \mu)$.

- c) In reality, \tilde{S} does of course depend on μ , and this have to be taken into account in the expression in c). The easiest way of doing this is by noting that \tilde{S} is multiplied with $e^{ik\mu(\eta-\eta_0)}$, and μ and $k(\eta-\eta_0)$ are therefore Fourier conjugate (just like k and x). This allows us to set

$$\mu \rightarrow \frac{1}{ik} \frac{d}{d\eta}$$

everywhere μ appears in \tilde{S} , just like we can set $ik \rightarrow d/dx$ in a standard Fourier transformation.

Use this to show that the full solution for the transfer function is

$$\Theta_l(\eta_0, k) = \int_0^{\eta_0} S(k, \eta) j_l[k(\eta_0 - \eta)] d\eta,$$

where

$$S(k, \eta) = e^{-\tau} \left[-\dot{\Phi} - \dot{\tau} \Theta_0 \right] + \frac{d}{d\eta} \left[e^{-\tau} \left(\Psi - \frac{v_b \dot{\tau}}{k} \right) \right]$$

(Hint: You may need your old knowledge about integration-by-parts to get this right :-))

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma_{\alpha\beta}^{\mu} = \frac{g^{\mu\nu}}{2} \left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right] \quad (8)$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^{\alpha} - \Gamma_{\mu\alpha,\nu}^{\alpha} + \Gamma_{\beta\alpha}^{\alpha} \Gamma_{\mu\nu}^{\beta} - \Gamma_{\beta\nu}^{\alpha} \Gamma_{\mu\alpha}^{\beta} \quad (9)$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu} \quad (10)$$

where $\mathcal{R} \equiv R_{\mu}^{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

- For a perfect fluid, the energy-momentum tensor is

$$T_{\nu}^{\mu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \quad (11)$$

where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four “time” variables: t = physical time, $\eta = \int_0^t a^{-1}(t) dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j = a^2(\eta) (-d\eta^2 + \delta_{ij} dx^i dx^j)$
- Friedmann’s equations:

$$H \equiv \frac{1}{a} \frac{da}{dt} = H_0 \sqrt{(\Omega_m + \Omega_b) a^{-3} + \Omega_r a^{-4} + \Omega_{\Lambda}} \quad (12)$$

$$\mathcal{H} \equiv \frac{1}{a} \frac{da}{d\eta} = H_0 \sqrt{(\Omega_m + \Omega_b) a^{-1} + \Omega_r a^{-2} + \Omega_{\Lambda} a^2} \quad (13)$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a' \mathcal{H}(a')} \quad (14)$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta'_0 = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \quad (15)$$

$$\Theta'_1 = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau' \left[\Theta_1 + \frac{1}{3}v_b \right], \quad (16)$$

$$\Theta'_l = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau' \left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2} \right], \quad l \geq 2 \quad (17)$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \quad l = l_{\max} \quad (18)$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \quad (19)$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \quad (20)$$

$$\delta'_b = \frac{k}{\mathcal{H}}v_b - 3\Phi' \quad (21)$$

$$v'_b = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau'R(3\Theta_1 + v_b) \quad (22)$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2} [\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0] \quad (23)$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2 a^2}\Omega_r\Theta_2 \quad (24)$$

1.4 Initial conditions

$$\Phi = 1 \quad (25)$$

$$\delta = \delta_b = \frac{3}{2}\Phi \quad (26)$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \quad (27)$$

$$\Theta_0 = \frac{1}{2}\Phi \quad (28)$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \quad (29)$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \quad (30)$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \quad (31)$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \quad (32)$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \quad (33)$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H} \tau' e^{-\tau(x)} = g(x) \quad (34)$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \quad (35)$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^0 \tilde{g}(x) dx = 1. \quad (36)$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b}, \quad (37)$$

where $n_b = \frac{\Omega_b \rho_c}{m_b a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\text{K}/a$, and $\epsilon_0 = 13.605698\text{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} [\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2], \quad (38)$$

where

$$C_r(T_b) = \frac{\Lambda_{2s \rightarrow 1s} + \Lambda_\alpha}{\Lambda_{2s \rightarrow 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \quad (39)$$

$$\Lambda_{2s \rightarrow 1s} = 8.227\text{s}^{-1} \quad (40)$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \quad (41)$$

$$n_{1s} = (1 - X_e) n_H \quad (42)$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \quad (43)$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b} \quad (44)$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27}\pi} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \quad (45)$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0/T_b) \quad (46)$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k, x) = \tilde{g} \left[\Theta_0 + \Psi + \frac{1}{4} \Theta_2 \right] + e^{-\tau} [\Psi' + \Phi'] - \frac{1}{k} \frac{d}{dx} (\mathcal{H} \tilde{g} v_b) + \frac{3}{4k^2} \frac{d}{dx} \left[\mathcal{H} \frac{d}{dx} (\mathcal{H} \tilde{g} \Theta_2) \right] \quad (47)$$

$$\frac{d}{dx} \left[\mathcal{H} \frac{d}{dx} (\mathcal{H} \tilde{g} \Theta_2) \right] = \frac{d(\mathcal{H} \mathcal{H}')}{dx} \tilde{g} \Theta_2 + 3 \mathcal{H} \mathcal{H}' (\tilde{g} \Theta_2 + \tilde{g} \Theta_2') + \mathcal{H}^2 (\tilde{g}'' \Theta_2 + 2 \tilde{g}' \Theta_2' + \tilde{g} \Theta_2''), \quad (48)$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}} \left[-\frac{\mathcal{H}'}{\mathcal{H}} \Theta_1 + \Theta_1' \right] + \frac{3}{10} [\tau'' \Theta_2 + \tau' \Theta_2'] - \frac{3k}{5\mathcal{H}} \left[-\frac{\mathcal{H}'}{\mathcal{H}} \Theta_3 + \Theta_3' \right] \quad (49)$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^0 \tilde{S}(k, x) j_l[k(\eta_0 - \eta(x))] dx \quad (50)$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0} \right)^{n-1} \Theta_l^2(k) \frac{dk}{k} \quad (51)$$