UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam for AST5220 — Cosmology II

Date: Tuesday, June 12th, 2013

Time: 09.00 – 13.00

The exam set consists of 9 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions

Answer each question with three or four sentences.

a) The geodesic equation reads

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

What are x^μ, λ and $\Gamma^\mu_{\alpha\beta}$ in this equation, and what does the geodesic equation describe?

Answer: $x^\mu = (t, x, y, z)$ is the four-position, λ is eigentime, and $\Gamma^{mu}_{\alpha\beta}$ is the Christoffel symbol. The geodesic equation describes straight lines in curved space, and therefore the path of free particles.

b) Why can we adopt the Boltzmann distribution to describe matter in this course?

Answer: Because the temperatures we consider in this course is much lower than the mass of even the lightest particles involved here, and annihilation and creation of new particles are therefore not taking place. For example, the mass of the electron is 511 keV, while the typical temperature during recombination is 13.6 eV.

c) Explain why cosmological perturbations do not grow until after $x \approx -10$, where $x = \ln a$.

Answer: Cosmological perturbations do not grow as long as their wavelengths are larger than the horizon at a given time. For the modes considered in this course, and that are relevant for the CMB power spectrum, the smallest modes start to enter the horizon around $x = -10$.

d) What is $\theta(x, t, \hat{p})$, and why is this a particularly important quantity in cosmological perturbation theory?

Answer: $\theta(x, t, \hat{p})$ is defined by $T(x, t, \hat{p}) = T_0(t)(1 + \theta(x, t, \hat{p})$, and is the photon temperature fluctuation observed from position x at time t in direction $\hat{p}$. The CMB field we observe today with experiments like WMAP and Planck is therefore $\theta(x = \mathbf{Earth}, t = textrmtoday, \hat{p})$, and this quantity therefore describes the most direct and powerful observable of the early universe in cosmology today.

e) When solving the Einstein-Boltzmann equations, we set $\Phi(a = 0) = 1$. Why is this acceptable?

Answer: This question has two parts. First, because our Einstein-Boltzmann equations are linear, their numerical solutions can be rescaled by any arbitrary constant after solving the equations. Second, the initial conditions for all other perturbation quantities are directly and linearly related to Φ_0. Therefore the exact value of Φ_0 may be chosen after solving the EB equations.

f) The power spectrum source function reads

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right]$$

Describe each term physically.

Answer: The first term is the Sachs-Wolfe contribution, which is generated at the last scattering surface ($\tilde{g}$) and consists of two components, namely the temperature fluctuation at a given position, Θ_0, and a corresponding gravitational potential, Ψ – the photons have to climb out of this potential to reach us, and therefore lose energy. The second term is the Integrated Sachs-Wolfe term: Photons gain or lose energy when moving through a time varying potential. The third term is the Doppler effect: Photons gain or lose energy due to peculiar velocities. Fourth and finally, the small terms involving Θ_2 are small polarization corrections.

Problem 2 – Physical interpretation

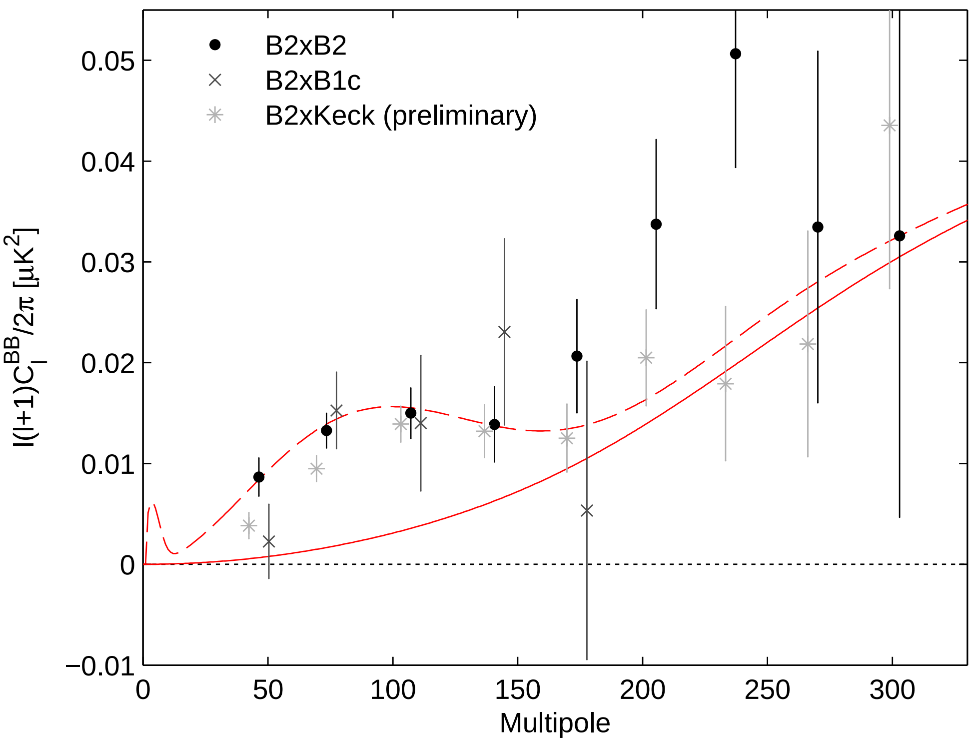

Figure 1: The BICEP2 B-mode power spectrum. The dashed red curve shows the best-fit theoretical spectrum, adopting the parameter values derived by the BICEP2 team.

Figure 1 shows the recently released BICEP2 B-mode polarization power spectrum, which led the BICEP2 team to claim a detection of inflationary gravitational waves with a tensor-to-scalar ratio of $r = 0.2^{+0.07}_{-0.05}$. The best-fit model including all effects are indicated by the dashed red line, and three different features may be seen in this curve, namely 1) a peak at $\ell \approx 10$, 2) a peak at $\ell \approx 100$, and 3) a rise toward high values.

a) Which physical effects cause/create each of the three features?

Answer: The first peak is caused by reionization; hot stars ionize the interstellar medium, creating very large scale fluctuations. The second peak are caused by gravitational waves at the last scattering surface. The rise at the high-ℓ end of the spectrum is gravitational lensing.

b) The three most important cosmological parameters for this multipole range are the amplitude of scalar perturbations, A_s, the optical depth of reionization, τ, and the tensor-to-scalar ratio, r. For each parameter, draw a cartoon of the theory spectrum, indicating what happens if you double its value.

Answer: Doubling A_s simply scales the entire spectrum up by a factor of two everywhere. Doubling τ increases the height of the first peak by roughly a

factor of four, while all other modes are slightly lower. Doubling r increases the amplitude of the first and second peak by a factor of two, but does not change the lensing part.

c) Explain why the optical depth of reionization, τ, is nearly degenerate with the spectral index of scalar perturbations, n_s.

Answer: Reionization adds power on the very largest scales, and smooths out structures on small scales. This corresponds to a negative tilt. It is therefore possible to partially counter-act an increase in τ by also increasing n_s, which is the tilt of the primordial power spectrum.

d) If one only has CMB temperature observations, τ is also almost perfectly degenerate with the amplitude of scalar perturbations, A_s. Explain why. Why do low-multipole polarization observations break this degeneracy?

Answer: By far the most important effect of τ on temperature observations is to decrease the high-ℓ spectrum; it is very difficult to measure the low-ℓ increase due to the dominant Sachs-Wolfe plateau, and large cosmic variance. An increase in τ can therefore be countered by an increase in A_s. However, this is not possible for polarization observations, because of the low-ℓ EE bump – this is a unique feature for reionization.

Problem 3 – The Boltzmann equation for CDM

To get the perturbations in an expanding universe right, one of the most central equations is the Boltzmann equation for cold dark matter (CDM). In this problem we will review some of the most important steps in relevant derivation, and we will work in the conformal Newtonian gauge,

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)(dx^2+dy^2+dz^2), \tag{1}$$

where Φ is the Newtonian potential and Φ is the curvature potential.

a) First, the symbolic Boltzmann equation is written in terms of a total time derivative, while it is computationally more convenient to work with partial derivatives in t, x, E and $\hat{p}$. Therefore, we first rewrite the Boltzmann equation in terms of partial derivatives:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\partial f}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t} = 0 \tag{2}$$

Why can we neglect the last term, depending on the direction of the momentum?

Answer: f is in our setting described by a Maxwell distribution, and this does not depend on the photon direction, $\hat{p}$, to zeroth order; it does so at least only to first order. However, in an unperturbed universe also $\mathrm{d}\hat{p}^i/\mathrm{d}t$ is zero to zeroth order, since photons move in straight lines unless acted upon by perturbations in the gravitational potential. The product of the two must therefore be at least second order, and can be neglected.

b) The energy of a massive particle is given by $E = \sqrt{p^2+m^2}$, and one also one knows from special relativity that $g_{\mu\nu}P^\mu P^\nu = -m^2$, where $P^\mu = (E, p^i)$ is the four-momentum of the particle. Show that $P^0 \approx E(1-\Psi)$.

Answer: See lecture notes.

c) Show that $P^i = \frac{p}{a}\hat{p}^i(1-\Phi)$.

Answer: See lecture notes.

d) Derive an expression for $\frac{\mathrm{d}x^i}{\mathrm{d}t}$ in terms of $\hat{p}$, p, E, a, Φ and Ψ. Physically, what does this equation tell us?

Answer: The equation looks like this,

$$\frac{dx^i}{dt} = \frac{p\hat{p}}{aE}(1+\Psi-\Phi), \tag{3}$$

and says that the peculiar velocity of a CDM particle decreases 1) when the universe expands (ie., a increases), and 2) when the particle moves out of an overdensity (ie., when Ψ, which is negative for overdensities, increases).

e) Finally, one has to compute $\frac{\mathrm{d}E}{\mathrm{d}t}$, which one can obtain from the geodesic equation – but, fortunately, we won't do that here. Instead, we simply write down the answer, namely

$$\frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial E}\left[\frac{p\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} + \frac{p^2}{E}H + \frac{p^2}{E}\frac{\partial \Phi}{\partial t}\right] = 0 \tag{4}$$

Then, we recall the definitions of the particle density and the mean velocity,

$$n = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f \tag{5}$$

$$v^i = \frac{1}{n} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f \frac{p\hat{p}^i}{E}, \tag{6}$$

$$\tag{7}$$

where $n = n^{(0)}(1+\delta)$. From this, derive the Boltzmann equation for the density of cold dark matter,

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0 \tag{8}$$

Answer: See lecture notes.

f) To close the system, we need actually two Boltzmann equations for CDM, because there are two unknown, δ and v. Outline schematically how one can obtain this second equation from equation 4.

Answer: Take the first moment of the equation, by first multiplying the original equation with $p\hat{p}/E$, and then integrate. This will give a differential equation for v^i instead of δ, and corresponds to conservation of momentum rather than mass.

Problem 4 – Christoffel symbols in the synchronous gauge

The synchronous gauge is defined through the following line element,

$$ds^2 = -dt^2 - a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j \tag{9}$$

where

$$h_{ij} = \begin{bmatrix} -2\phi & 0 & 0 \\ 0 & -2\phi & 0 \\ 0 & 0 & h+4\phi \end{bmatrix}, \tag{10}$$

and $\phi = \phi(x,t)$, $h = h(x,t)$ are two perturbation fields. Compute Γ^0_{00}, Γ^0_{03}, Γ^0_{11}, Γ^0_{23}, Γ^0_{33}, Γ^1_{11}, Γ^2_{10}, Γ^3_{22}, and Γ^3_{33}. (PS! The general expression for the Christoffel symbols is given in the Appendix. PPS! For those of you who thinks this is a lot of work – the first version of this problem requested calculation of all 64 symbols. Consider yourself lucky!)

Answer:

$$\Gamma^0_{00} = \Gamma^0_{03} = 0 \tag{11}$$

$$\Gamma^0_{11} = a^2(-H + 2H\phi + \dot{\phi}) \tag{12}$$

$$\Gamma^0_{23} = 0 \tag{13}$$

$$\Gamma^0_{33} = -a^2(H(1+h+4\phi) + (\dot{h} + 4\dot{\phi})) \tag{14}$$

$$\Gamma^1_{11} = \frac{\partial\phi}{\partial x} \tag{15}$$

$$\Gamma^2_{10} = 0 \tag{16}$$

$$\Gamma^3_{22} = \frac{\partial\phi}{\partial z} \tag{17}$$

$$\Gamma^3_{33} = \frac{1}{2}\frac{\partial h}{\partial z} + 2\frac{\partial\phi}{\partial z} \tag{18}$$

$$\tag{19}$$

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{20}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{21}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{22}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.
- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{23}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{24}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{25}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{26}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{27}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{28}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{29}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{30}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{31}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{32}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{33}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{34}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{35}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{36}$$

1.4 Initial conditions

$$\Phi = 1 \tag{37}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{38}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{39}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{40}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{41}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{42}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{43}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{44}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{45}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{46}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{47}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{48}$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b}, \tag{49}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \tag{50}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\rightarrow 1s} + \Lambda_\alpha}{\Lambda_{2s\rightarrow 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{51}$$

$$\Lambda_{2s\rightarrow 1s} = 8.227\mathrm{s}^{-1} \tag{52}$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{53}$$

$$n_{1s} = (1 - X_e) n_H \tag{54}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0 / 4T_b} \tag{55}$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0 / T_b} \tag{56}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \tag{57}$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0 / T_b) \tag{58}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{59}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{60}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{61}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{62}$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{63}$$