UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220 — Cosmology II

Date: Tuesday, June 4th, 2013

Time: 09.00 – 13.00

The exam set consists of 13 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions

Answer each question with three or four sentences.

a) Why are tensor equations important in General Relativity? **Answer: Only tensor equations are equally valid in all coordinate systems, and therefore it must be possible to write any true physical law as a tensor equation.**

b) What do the Fermi-Dirac and Bose-Einstein distributions describe? What does the Maxwell distribution describe? **Answer: The Fermi-Dirac distribution describes the distribution function for fermions (spin-half particles, such as electrons, protons etc.), while the Bose-Einstein describes the distribution function for bosons (spin-integer particles, such as photons, gravitons etc.) The Maxwell distribution is the low-temperature limiting case of both distributions, and the distribution of choice for most of cosmology.**

c) The Einstein equation for tensor perturbations reads

$$\ddot{h} + 2\frac{\dot{a}}{a}h + k^2 h = 0. \tag{1}$$

What sort of an equation is this, and how can it be quantized? **Answer: This is a damped harmonic oscillator. To quantize it, introduce the change-of-variable** $\tilde{h} = ah$**, which brings it to a perfect harmonic oscillator, which is quantized for instance with raising and lowering operator.**

d) Why is it acceptable to set the curvature potential, Φ, at early times to 1 at early times when solving the Einstein-Boltzmann equations? **Answer: Because the Einstein-Boltzmann equations are linear. Normalization to a physical power spectrum, $P(k)$, can be done at the very end, after computing the effective transfer function.**

e) When deriving the Boltzmann equation for photons, one finds that

$$\frac{dx^i}{dt} \approx \frac{\hat{p}}{a}(1 - \Phi + \Psi), \tag{2}$$

where (t, x^i) is the photon position four-vector, $\hat{p}$ is the photon direction, a is the scale factor, and Φ and Ψ are the curvature and Newtonian potentials, respectively. What is the physical interpretation of this equation? **Answer: The left-hand side describes the "velocity" of the photon, which really is the momentum of the photon, since the speed is actually fixed. So, this equation says that a photon loses momentum when either a increases (ie., the universe expands) or Ψ decreases (ie., the photon moves into an overdensity).**

f) What is the main advantage of the CMB power spectrum line-of-sight integration method? **Answer: Computational cost. Instead of having to solve a set of 2000 coupled equations, we only have to solve a set of 15 equations, and then perform a few one- or two-dimensional integrals.**

Problem 2 – Physical interpretation

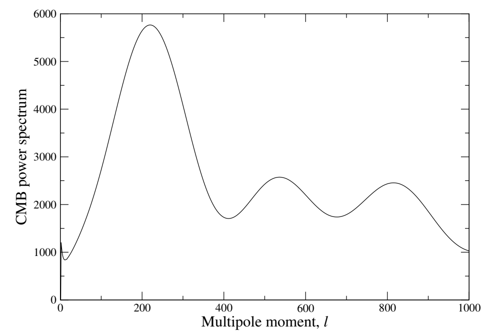

Figure 1: The best-fit LCDM WMAP temperature power spectrum.

Figure 1 shows the best-fit LCDM WMAP power spectrum as computed from the 7-year data release. In the following we will consider how this compares to the more recent Planck release in terms of cosmological parameters:

a) The WMAP best-fit value of the spectral index of scalar perturbations, $n_{\rm s}$, is 0.972 ± 0.013, while Planck finds 0.9616 ± 0.0094. How does this difference manifest itself in the power spectrum? Draw a cartoon illustrating the difference. What physical mechanism determines the value of n_s? **Answer: When the power spectrum becomes negatively tilted, there is relatively more power on large than on small angular scales. The numerical value of n_s is determined by inflation, and in particular the end of the inflation.**

b) The best-fit WMAP value of the baryon density is $\Omega_b = 0.0463$, while Planck finds $\Omega_b = 0.049$. How is this seen in the observed power spectrum? Draw a cartoon to illustrate this. Explain this effect physically. **Answer: A high value of Ω_b means much baryonic matter. This leads to accoustic compressions being stronger than rarefications, since the extra baryons now starts to contribute themselves to the potential during compressions. This implies that the first and third peak in the spectrum increases relative to the second and fourth peak.**

c) How does the optical depth, τ, affect the power spectrum at low and high multipoles? Explain why there is a strong degeneracy between the power spectrum amplitude, A, and the optical depth. **Answer: The optical depth to reionization is caused by early heavy stars outputting large amount of ultraviolet photons, which ionize the neutral hydrogen around the stars. At first this happen in patches, tracing where the first stars are formed. However, eventually the whole universe become ionized. This implies that extra power is generated on large scales, due to the patchiness, while the main effect on small scales is that structures are "washed" out by a "shower door" effect – the direction of the photons are changed by the rescattering. Effectively, one observes an effect on the power spectrum resembling a negative tilt, in that large scales are slightly boosted, while all others (above, say, $\ell = 50$–100) are suppressed. This also implies an overall degeneracy with the amplitude, since an increase in τ can be partially countered with a corresponding increase in A. The lowest ℓ's**

contribute little to the overall value of A due to cosmic variance, and the high-ℓ effect of τ is a nearly constant multiplicative factor.

d) As the first experiment ever, Planck has sufficient angular resolution and sensitivity to exploit gravitational lensing for parameter estimation. This effect is due to CMB photons being bent by the presence of dark matter fluctuations between the last-scattering surface and ourselves, such that the photon appear to us to come from a slightly different direction than its true point of origin. How is the high-l power spectrum changed by these distortions, and how does this effect break the A-τ degeneracy? **Answer: Lensing implies a slight transfer or leakage of power across angular scales, such that the peaks are slightly lower and troughs are slightly higher with lensing than without. Further, the stronger the peak smoothing effect is, the stronger the lensing is. And a strong lensing effect implies large matter fluctuations – and therefore a strong power amplitude. Therefore, by measuring how much the peaks are smoothed out, we get a direct and second measurement of the power amplitude, independent of the optical depth.**

Problem 3 – The Saha equation

The Saha equation plays a central role when calculating the CMB power spectrum. In the following, we will derive one form of this equation suitable for this purpose.

For a gas consisting of photons, protons and electrons in thermodynamic equilibrium, one can show that the following relation holds

$$\frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} = \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}},$$

where n_X is the density of species X, and

$$n_X^{(0)} = \int \frac{d^3 p}{(2\pi\hbar)^3} e^{-\frac{E_X}{kT}}$$

is the equilibrium density. (Here, p denotes momentum of the particle, $E_X = \sqrt{p_X^2 c^2 + m_X^2 c^4}$ is the energy, and T is the common equilibrium temperature.) We will only consider cases for which $mc^2 >> kT$, ie., systems for which the temperature is much lower than the rest mass of the particles.

Next, define the free electron fraction to be

$$X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}, \tag{3}$$

where the latter equality holds due to the requirement of a neutral universe.

Finally, you can assume as known that the photon density equals the equilibrium density during thermodynamic equilibrium, $n_\gamma = n_\gamma^{(0)}$.

a) Show that the Saha equation may be written on the form

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

Answer: See textbook.

b) Show that the background density of (massive) species X is given by

$$n_X^{(0)} = \left(\frac{kTm_X}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{m_X c^2}{kT}}.$$

Hint: You may need to know that

$$\int_0^\infty \sqrt{u}e^{-u}du = \frac{\sqrt{\pi}}{2}.$$

Answer: See textbook.

c) Finally, show that the full Saha equation for the electron density is given by

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_e+n_H}\left(\frac{kTm_e}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{\epsilon_0}{kT}}.$$

What is ϵ_0 here? Which assumption regarding m_p and m_H is used here? **Answer: See textbook; ϵ_0 is the hydrogen binding energy. We assume that $m_p \approx m_H$ for the multiplicative part, but not for the exponential part.**

d) Why can't we use the Saha equation at all times, but must instead use the Peebles equation at late times? **Answer: The Saha equation assumes thermodynamic equilibrium at all times. However, this is no longer true when the density of the universe have become sufficiently low that it is hard for a proton to find an electron to combine with. To account for this effect, we need the full Peebles equation.**

Problem 4 – Initial conditions

In order to solve the Boltzmann-Einstein equations, one needs initial conditions, and in this problem you will derive the appropriate expressions for Ψ, δ, δ_b, v, v_b and Θ_0, relating all to Φ. You should neglect neutrinos and polarization in this problem, and you need only consider adiabatic initial conditions.

a) Starting from the full set of perturbation equations listed in the Appendix, write down a simplified set that is valid for very early times. Why can terms containing factors of $k/\mathcal{H}$ be neglected? **Answer: See textbook. Since $\mathcal{H} \sim 1/\eta$, we have that $k/\mathcal{H} \sim k\eta$. Since k is fixed, but the initial time is free to vary, it is always possible to choose a sufficiently small conformal time to make sure that $k\eta << 1$, in which case these can be neglected compared to the other terms.**

b) Derive a second-order differential equation for Φ as a function of a. Show that this equation has two solutions. Which solution survives and is therefore cosmologically relevant? **Answer: See textbook. The equation has two power-law solutions, one with $p = 0$ and one with $p = -3$. Only the $p = 0$ (constant) solution survives, to produce the initial perturbations we observe today.**

c) Find the appropriate initial conditions for Θ_0, δ and δ_b expressed relative to Φ. **Answer: See textbook; $\Theta_0 = 0.5\Phi$, $\delta = \delta_b = 1.5\Phi$.**

d) Why is $\Theta_3 << \Theta_2 << \Theta_1 << \Theta_0$ at early times? **An-**

swer: At early times, the optical depth is very large, and a photon cannot move effectively more than a few centimeters. Inside this small volume, the temperature is very nearly constant, and the dominant term is the monopole. However, the dipole gets a non-negligible contribution from the Doppler effect, due to internal motions in the plasma. After that, each higher-order term falls proportionally with an additional factor of τ, effectively washing out any higher-order structure.

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{4}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{5}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{6}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{7}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{8}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{9}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{10}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{11}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{12}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{13}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{14}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{15}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{16}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{17}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{18}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{19}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{20}$$

1.4 Initial conditions

$$\Phi = 1 \tag{21}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{22}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{23}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{24}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{25}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{26}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{27}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{28}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{29}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{30}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{31}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{32}$$

- The Saha equation,

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{33}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\text{K}/a$, and $\epsilon_0 = 13.605698\text{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b}\left[\beta(T_b)(1-X_e) - n_H \alpha^{(2)}(T_b) X_e^2\right], \tag{34}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\rightarrow 1s} + \Lambda_\alpha}{\Lambda_{2s\rightarrow 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{35}$$

$$\Lambda_{2s\rightarrow 1s} = 8.227\text{s}^{-1} \tag{36}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{37}$$

$$n_{1s} = (1-X_e)n_H \tag{38}$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \tag{39}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{40}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{41}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b) \tag{42}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{43}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{44}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{45}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{46}$$

3. The CMB spectrum:

$$C_l = \int_0^{\infty} \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{47}$$