UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220 — Cosmology II *with answers*

Date: Wednesday, June 14th, 2012

Time: 09.00 – 13.00

The exam set consists of 15 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions

Answer each question with three or four sentences.

a) Write down the Boltzmann equation on schematic form, and expand the left-hand side into partial derivatives using x, p, $\hat{p}$ and t as free variables. Why can we neglect the term depending on $\hat{p}$?

Answer:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}}\frac{d\hat{p}}{dt} = C[f] \tag{1}$$

We can neglect $\frac{\partial f}{\partial \hat{p}}\frac{d\hat{p}}{dt}$ because both factor individually are first-order, and the product is therefore second order: $\frac{\partial f}{\partial \hat{p}}$ is first-order because the Maxwell distribution does not depend on the direction of the photon momentum, while $\frac{d\hat{p}}{dt}$ is first-order because if there are no perturbations, then the photon will continue moving in the same direction.

b) What is the physical interpretation of the conformal time, η?

Answer: The conformal time is the distance light has been able to travel since the Big Bang. It is therefore equal to the horizon.

c) What is the main difference between dark matter and baryons?

Answer: Dark matter does not interact with light, and is pressureless. Dark matter clumps less than baryons for the same reason.

d) How does inflation solve the so-called isotropy problem?

Answer: Inflation expands the size of the universe by $\sim 10^{28}$ in $\sim 10^{-34}$ seconds. That means we only need to establish thermodynamic equilibrium with a region that is 10^{28} times smaller in order to see a uniform temperature distribution *after* inflation, than if we did not have inflation. And that is easy, since our current observable universe completely fits inside a small sphere that is much smaller than the horizon prior to inflation, when taking into account this factor.

e) During tight coupling we neglect all photon moments except Θ_0, Θ_1 and Θ_2. Why can't we neglect also Θ_1 and Θ_2?

Answer: Θ_1 is non-neglible because of the peculiar baryon motion, v_b, which creates a local dipole due to the Doppler effect. Θ_2 is non-negligible because of polarization, which does not couple either to Θ_0 or Θ_1.

f) Why is the position of the first peak in the CMB power spectrum a sensitive probe of the total density of the universe? (Explain with a drawing if you think that is useful.)

Answer: The physical size of a CMB hot spot is given by the plasma physics at the time of recombination, and is independent of what happens to the universe later. However, photons move along null-geodesics, and in a closed universe, these converge; in a flat universe they are in fact straight lines; while in an open universe, they diverge. Thus,

the apparent size of a CMB as seen today depends on the geometry of the universe, and therefore the density.

Problem 2 – Recombination and the electron fraction

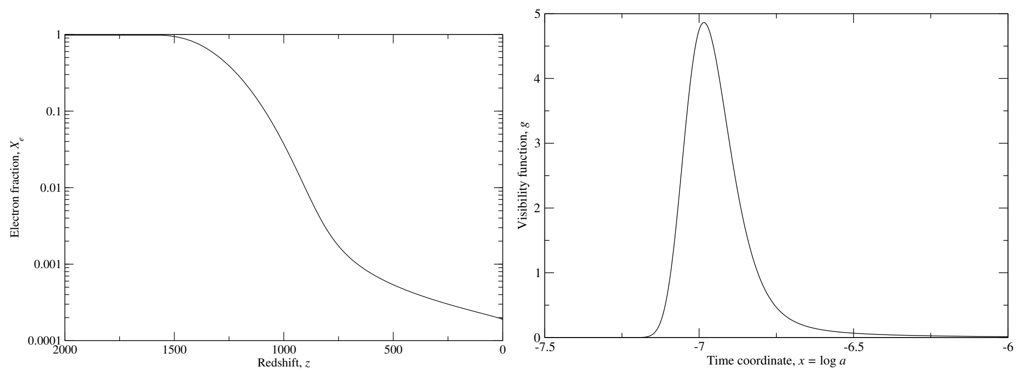

Figure 1: Left: The electron fraction as a function of redshift. Right: The visibility function, g.

The CMB field we observe today is to a very large extent created during the recombination epoch, when electrons and protons combined into neutral hydrogen. In order to understand this period quantitatively, it is necessary to know the electron fraction as a function of time, and we have in fact computed this during the course work, and the result is shown in the left panel of Figure 1. The right panel shows the corresponding visibility function, g.

a) As seen in the plot of X_e above, it is possible to define more or less three redshift ranges for the electron fraction, namely $z > 1500$, $1500 > z > 750$ and $z < 750$. Explain this behaviour.

Answer: At $z > 1500$ the temperature of the universe was higher than 3000K, and filled with photons with energy above the ionization energy of hydrogen. As a result, any neutral hydrogen atom

would instantly be hit by an ionizing photon, effectively maintaining $X_e = 1$. However, when the tmperature fell below 3000K, neutral hydrogen was able to form, binding up free electrons. This is why X_e starts to fall after $z = 1500$. This process keeps happening until the expansion rate of the universe becomes significant, making the mean distance between free particles so large that they don't find eachother; even though there are free partners out there, it becomes difficult to find them. This is what happens after $z = 750$, and the result is a flattening of X_e, and the residual electron population freezes in.

b) As long as $X_e > 0.99$ we used the Saha equation

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b}, \tag{2}$$

to solve for X_e, which basically is the Boltzmann equation applied to the process $e + p \leftrightarrow H + \gamma$. But it does not apply always. What requirement(s) must be fulfilled in order for the Saha equations to hold?

Answer: The Saha equation holds only for 1) thermodynamic equilibrium, and when 2) the expansion rate of the universe is significantly smaller than the recombination reaction rate.

c) At some stage one has to switch from the Saha equation the more accurate Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right]. \tag{3}$$

This describes recombination to the first excited hydrogen state ($n = 2$), not to the ground state ($n = 1$). Why is recombination to the ground state not relevant in our case?

Answer: Recombination to the ground state emits a photon with energy of 13.6 eV, which instantly will be captured by another free hydrogen atom. The net number of free electrons is therefore not changed by this reaction.

d) What is the physical interpretation of the visibility function? Why is it zero at $x < -8$? Why is it zero at $x > -5$?

Answer: The visibility function is the probability for a given photon to have been *last scattered* at redshift z. It is zero for $x < -8$ because the density is so high that that photon is certain to be scattered multiple times after this. It is zero for $x > -5$ because there are no free electrons to scatter with anymore.

Problem 3 – The Einstein equations

One of the most central parts of AST5220 is to derive and solve the linear Boltzmann-Einstein equations. In this problem we will therefore derive the first Einstein equation, precisely like we did in the lectures.

Before we start the real work, we have to choose a gauge, and we decided to adopt the conformal Newtonian gauge for our studies,

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)(dx^2+dy^2+dz^2). \quad (4)$$

Here Φ is the Newtonian potential and Φ is the curvature potential.

a) Given this metric, the first step is to derive the Christoffel symbols. The only non-zero Christoffel symbols are

$$\Gamma^0_{00} = \Psi_{,0} \quad (5)$$
$$\Gamma^0_{0i} = \Gamma^0_{i0} = ik_i\Psi \quad (6)$$
$$\Gamma^0_{ij} = \delta_{ij}a^2[H + 2H(\Phi - \Psi) + \Phi_{,0}] \quad (7)$$
$$\Gamma^i_{00} = \frac{ik^i}{a^2}\Psi \quad (8)$$
$$\Gamma^i_{j0} = \Gamma^i_{0j} = \delta_{ij}(H + \Phi_{,0}) \quad (9)$$
$$\Gamma^i_{jk} = i\Phi[\delta_{ij}k_k + \delta_{ik}k_j - \delta_{jk}k_i] \quad (10)$$

Derive the expressions for Γ^0_{0i} and Γ^i_{jk}.

Answer: See Dodelson or Øystein's lecture notes.

b) The next step is to compute the Ricci tensor, which in general reads

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}. \quad (11)$$

The $0i$-component of this tensor is zero, while the ij-component is

$$R_{ij} = \delta_{ij}[(2a^2H^2 + a\ddot{a})(1 + 2\Phi - 2\Psi) + a^2H(6\Phi_{,0} - \Psi_{,0}) + a^2\Phi_{,00} + k^2\Phi] + k_ik_j(\Phi + \Psi).$$

But what is R_{00}? As a help on your way, I will let you know that

$$\Gamma^i_{00}, i = \frac{-k^2}{a^2}\Psi, \qquad \Gamma^i_{i\beta}\Gamma^\beta_{00} = \Gamma^i_{i0}\Gamma^0_{00} = 3H\Psi_{,0}. \qquad (12)$$

You do not have to show this.

Answer: See Dodelson or Øystein's lecture notes.

c) Third, we have to compute the Ricci scalar, $\mathcal{R} \equiv R^\mu_\mu = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij}$. This is rather ugly, since there are quite a lot of terms involved, and we won't spend time on it here. Instead, we just write down the final expression, including only first-order terms,

$$\delta\mathcal{R} = -12\Psi(H^2 + \frac{\ddot{a}}{a}) + \frac{2k^2}{a^2}\Psi + 6\Phi_{,00} - 6H(\Psi_{,0} - 4\Phi_{,0}) + 4\frac{k^2\Phi}{a^2}$$

Using this expression and the results derived above, show that the first-order contribution to the Einstein tensor is

$$\delta G^0_0 = -6H\Phi, 0 + 6\Psi H^2 - 2\frac{k^2\Phi}{a^2} \qquad (13)$$

Answer: See Dodelson or Øystein's lecture notes.

d) To complete the Einstein equation,

$$\delta G^0_0 = 8\pi\delta T^0_0$$

we need an energy-momentum tensor on the right-hand side. You will not be asked to derive this, but only describe qualitatively what goes into it: Which components are the bare minimum we need to include in order to obtain a physically relevant power spectrum, ie., one that looks qualitatively similar to the current ΛCDM spectrum? Which of these components was most important at very early times, at $x = \log a \sim -10$, and why?

Answer: The energy-momentum tensor describes the content of the universe, both in terms of energy and pressure. The absolute minimum set of components we need to obtain a reasonable power spectrum is 1) dark matter, 2) baryons and 3) photons. In addition we should add at least 4) neutrinos, but that is a relatively small correction factor compared to the others. Photons is the most important one at early times, because the energy density of photons fall as a^{-4}, while it scales as a^{-3} for dark matter and baryons. The cosmological constant does not kick in until late times, as it is independent of a.

Problem 4 – Line-of-sight integration

A crucial part in speeding up a Boltzmann code is to change from direct integration of the Boltzmann-Einstein equations to a smarter approach, called "line-of-sight" integration. In this problem, we will derive the expression for the transfer function $\Theta_l(k)$ in terms of the source function.

Before we begin, let us review some relations concerning the Legendre polynomials, $P_l(\mu)$, that you may or may not find useful in the following:

$$
\begin{aligned}
P_0(\mu) &= 1 \\
P_1(\mu) &= \mu \\
P_l(\mu) &= (-1)^l P_l(-\mu) \\
\int_{-1}^{1} P_l(\mu) P_{l'}(\mu) d\mu &= \delta_{ll'} \frac{2}{2l+1} \\
j_l(x) &= \frac{1}{2i^l} \int_{-1}^{1} e^{i\mu x} P_l(\mu) d\mu \\
f_l &= \frac{i^l}{2} \int_{-1}^{1} f(\mu) P_l(\mu) d\mu
\end{aligned}
$$

Here $j_l(x)$ is the spherical Bessel function of order l, and $f(\mu)$ is an arbitrary function defined between -1 and 1.

Also, note that in the following, $\dot{}$ means derivative with respect to conformal time.

a) First, from an coding point of view, why does one obtain such a large speed-up with the line-of-sight integration approach compared to direct integration?

Answer: With direct integration one has to solve for thousands of Θ_l's; with line-of-sight integration one only needs to solve for 6-10.

b) In a few sentences, explain what the main physical difference is between the line-of-sight integration and the direct solution approaches.

Answer: The main difference is that in the direct integration approach, one first decomposes $\Theta(k, \eta, \hat{p})$ into multipole moments, and then solves the Einstein-Boltzmann equations; in the line-of-sight integration approach, one first solves the equations, and

then expands into multipole moments. Physically speaking, the line-of-sight approach only requires knowledge about local physics, while the brute-force approach requires knowledge about global physics.

c) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

where $\Theta = \Theta(k, \mu, \eta)$ and μ is the angle between the photon propagation direction, $\hat{p}$, and the wave vector, $\hat{k}$. Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}[\Theta_0 + \mu v_b],$$

and show that equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S} e^{ik\mu(\eta - \eta_0) - \tau} d\eta.$$

(Note that we have dropped a quadrupole/polarization term in this expression, in order to keep things simple(r).)

Answer: See Dodelson

d) Assume that $\tilde{S}$ does not depend on μ (in this sub-problem only). Show that in this case

$$\Theta_l(\eta_0, k) = (-1)^l \int_0^{\eta_0} \tilde{S} e^{-\tau} j_l[k(\eta - \eta_0)] d\eta,$$

where $\Theta_l(\eta, k)$ are the multipole expansion coefficients of $\Theta(\eta, k, \mu)$.

Answer: See Dodelson

e) (*Hint: This sub-problem is the toughest in the exam set. Don't spend all your time on getting this right, but rather do it after finishing up the other problems.*) In reality, $\tilde{S}$ does of course depend on μ, and this have to be taken into account in the expression in c). The easiest way of doing this is by noting that $\tilde{S}$ is multiplied with $e^{ik\mu(\eta-\eta_0)}$, and μ and $k(\eta - \eta_0)$ are therefore Fourier conjugate (just like k and x). This allows us to set

$$\mu \rightarrow \frac{1}{ik}\frac{d}{d\eta}$$

everywhere μ appears in $\tilde{S}$, just like we can set $ik \rightarrow d/dx$ in a standard Fourier transformation.

Use this to show that the full solution for the transfer function is

$$\Theta_l(\eta_0, k) = \int_0^{\eta_0} S(k, \eta) j_l[k(\eta_0 - \eta)] d\eta,$$

where

$$S(k, \eta) = e^{-\tau}\left[-\dot{\Phi} - \dot{\tau}\Theta_0\right] + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{v_b\dot{\tau}}{k}\right)\right]$$

(Hint: You may need your old knowledge about integration-by-parts to get this right :-))

Answer: See Dodelson

f) Introducing the visibility function, $g(x) \equiv -\dot{\tau}e^{-\tau}$, the source function can be rewritten into the form

$$S(k, \eta) = g[\Theta_0 + \Psi] + \frac{d}{d\eta}\left(\frac{gv_b}{k}\right) + e^{-\tau}\left[\dot{\Psi} - \dot{\Phi}\right].$$

What is the physical interpretation of each of these terms?

Answer: The first term is the Sachs-Wolfe contribution, which consists of two parts. First, the physical temperature (Θ_0) at a given position defines the basic wavelength of the emitted photons. However, these photons live in a gravitational potential, Ψ, and are either redshifted or blueshifted depending on whether the current position is a over- or underdensity. Ψ contributes in fact with a larger effect than Θ_0, and so the photons emerging from an overdensity actually actually have longer wavelength than those emerging from an underdensity. Note that this term is multiplied with the visibility function, g, which reveals that this effect originates from the last-scattering surface.

The second term is the Doppler effect, which depends on the local baryon velocity, v_b. This is also multiplied with g, and arises on the last-scattering surface.

The last term is the integrated Sachs-Wolfe term, which is caused by *changes* in the gravitational potential as the photons move through space. If the photon has fallen into an overdensity, it gains energy. If the overdensity decays while the photon is on its way out, it will not lose as much energy as it gained on its way in. Note that this term is not multiplied with g, and is not an effect localized to the last-scattering surface.

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma^{\mu}_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right] \tag{14}$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \tag{15}$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu} \tag{16}$$

  where $\mathcal{R} \equiv R^{\mu}_{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

- For a perfect fluid, the energy-momentum tensor is

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{17}$$

  where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four "time" variables: t = physical time, $\eta = \int_0^t a^{-1}(t)dt$ = conformal time, a = scale factor, $x = \ln a$

- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$

- Friedmann's equations:

$$H \equiv \frac{1}{a}\frac{da}{dt} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \tag{18}$$

$$\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\eta} = H_0\sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \tag{19}$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a'\mathcal{H}(a')} \tag{20}$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta_0' = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \tag{21}$$

$$\Theta_1' = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \tag{22}$$

$$\Theta_l' = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau'\left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2}\right], \qquad l \geq 2 \tag{23}$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \qquad l = l_{\max} \tag{24}$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \tag{25}$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \tag{26}$$

$$\delta_b' = \frac{k}{\mathcal{H}}v_b - 3\Phi' \tag{27}$$

$$v_b' = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b) \tag{28}$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{29}$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2a^2}\Omega_r\Theta_2 \tag{30}$$

1.4 Initial conditions

$$\Phi = 1 \tag{31}$$

$$\delta = \delta_b = \frac{3}{2}\Phi \tag{32}$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \tag{33}$$

$$\Theta_0 = \frac{1}{2}\Phi \tag{34}$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \tag{35}$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \tag{36}$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \tag{37}$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \tag{38}$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \tag{39}$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \tag{40}$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \tag{41}$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^{0} \tilde{g}(x) dx = 1. \tag{42}$$

- The Saha equation,

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}, \tag{43}$$

where $n_b = \frac{\Omega_b \rho_c}{m_h a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\mathrm{K}/a$, and $\epsilon_0 = 13.605698\mathrm{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b}\left[\beta(T_b)(1-X_e) - n_H \alpha^{(2)}(T_b) X_e^2\right], \tag{44}$$

where

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \tag{45}$$

$$\Lambda_{2s\to 1s} = 8.227\mathrm{s}^{-1} \tag{46}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{47}$$

$$n_{1s} = (1-X_e)n_H \tag{48}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b} \tag{49}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b} \tag{50}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{51}$$

$$\phi_2(T_b) = 0.448\ln(\epsilon_0/T_b) \tag{52}$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2\right] + e^{-\tau}\left[\Psi' + \Phi'\right] - \frac{1}{k}\frac{d}{dx}(\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] \tag{53}$$

$$\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Theta_2)\right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx}\tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \tag{54}$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1'\right] + \frac{3}{10}\left[\tau''\Theta_2 + \tau'\Theta_2'\right] - \frac{3k}{5\mathcal{H}}\left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3'\right] \tag{55}$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^{0} \tilde{S}(k,x) j_l[k(\eta_0 - \eta(x))]dx \tag{56}$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0}\right)^{n-1} \Theta_l^2(k)\frac{dk}{k} \tag{57}$$