

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam for AST5220/9420 — Cosmology II

Date: Friday, June 11th, 2010

Time: 09.00 – 12.00

The exam set consists of 16 pages.

Appendix: Equation summary

Allowed aids: None.

Please check that the exam set is complete before answering the questions. Note that AST5220 students are supposed to answer problems 1)-4), while AST9420 students answer problems 1)-3) and 5). Each problem counts for 25% of the final score. Note that the exam may be answered in either Norwegian or English, even though the text is in English.

Problem 1 – Background questions (AST5220 and AST9420)

Answer each question with three or four sentences.

- a) The metric for the conformal Newtonian gauge is

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)(dx^2 + dy^2 + dz^2).$$

What is the physical interpretation of Ψ and Φ ? At late times, how is Ψ numerically related to Φ ?

- b) The geodesic equation reads

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$$

What are x^μ , λ and $\Gamma_{\alpha\beta}^\mu$ in this equation? What does the geodesic equation describe?

- c) Write down the Boltzmann equation on schematic form. Which physical effects must be included in the collision term for 1) cold dark matter and 2) baryonic matter, respectively?
- d) What is the optical depth, τ ? Why is $\tau \approx 1$ a special value?
- e) What is the main advantage of solving the Boltzmann-Einstein in Fourier space instead of in real-space?
- f) Briefly list the (up to four) main motivations for introducing inflation in cosmology. (No more than two sentences per item.)

Problem 2 – Physical interpretation (AST5220 and AST9420)

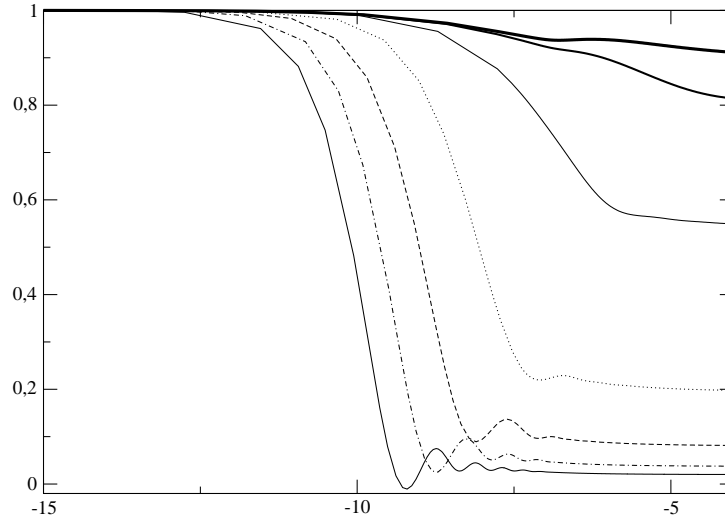


Figure 1: The function Φ is plotted for a few different values of the wavenumber k , as function of time, x .

Figure 1 shows the curvature potential, Φ , which enters into the solution of the Einstein-Boltzmann equations, plotted for a few different values of k (between 0.1 and $1000 H_0/c$) and as a function of time, x , where $x = \ln a$.

- Which main phases can you see here?
- Explain physically the general behaviour of these functions. Why are the functions flat after $x \approx -6$? In the standard Λ CDM model (ie., $\Omega_b \approx 0.05$, $\Omega_m \approx 0.25$ and $\Omega_\Lambda \approx 0.7$), will they remain flat to $x = 0$? If not, why?
- Would the late-time constant plateau for small-scale modes shift to higher or lower values (up or down) if we double Ω_m ? Why?

Problem 3 – The Saha equation (AST5220 and AST9420)

The Saha equation plays a central role when calculating the CMB power spectrum. In the following, we will derive one form of this equation suitable for this purpose.

For a gas consisting of photons, protons and electrons in thermodynamic equilibrium, one can show that the following relation holds

$$\frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} = \frac{n_H n_\gamma}{n_H^{(0)} n_\gamma^{(0)}},$$

where n_X is the density of species X , and

$$n_X^{(0)} = \int \frac{d^3 p}{(2\pi\hbar)^3} e^{-\frac{E_X}{kT}}$$

is the equilibrium density. (Here, p denotes momentum of the particle, $E_X = \sqrt{p_X^2 c^2 + m_X^2 c^4}$ is the energy, and T is the common equilibrium temperature.) We will only consider cases for which $mc^2 \gg kT$, i.e., systems for which the temperature is much lower than the rest mass of the particles.

Next, define the free electron fraction to be

$$X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}, \quad (1)$$

where the latter equality holds due to the requirement of a neutral universe.

Finally, you can assume as known that the photon density equals the equilibrium density during thermodynamic equilibrium, $n_\gamma = n_\gamma^{(0)}$.

a) Show that the Saha equation may be written on the form

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}}$$

- b) Show that the background density of (massive) species X is given by

$$n_X^{(0)} = \left(\frac{kTm_X}{2\pi\hbar^2} \right)^{3/2} e^{-\frac{m_X c^2}{kT}}.$$

Hint: You may need to know that

$$\int_0^\infty \sqrt{u} e^{-u} du = \frac{\sqrt{\pi}}{2}.$$

- c) Finally, show that the full Saha equation for the electron density is given by

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_e + n_H} \left(\frac{kTm_e}{2\pi\hbar^2} \right)^{3/2} e^{-\frac{\epsilon_0}{kT}}.$$

What is ϵ_0 here? Which assumption regarding m_p and m_H is used here?

- d) Why can't we use the Saha equation at all times, but must instead use the Peebles equation at late times?

Problem 4 – The Boltzmann equation for free photons (AST5220)

In this problem you will derive the Boltzmann equation for free photons, ie., neglecting any collision terms, in the conformal Newtonian gauge,

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)(dx^2 + dy^2 + dz^2).$$

- a) For a massless photon one has $g_{\mu\nu}P^\mu P^\nu = 0$, where $P^\mu = dx^\mu/d\lambda = (E, \vec{p})$ is the four-momentum of the photon. Define $p^2 = g_{ij}P^i P^j$. Show that

$$\begin{aligned} P^0 &= p(1 - \Psi) \\ P^i &= \frac{p}{a}\hat{p}^i(1 - \Phi) \end{aligned}$$

to first order in Ψ and Φ , where \hat{p} is the unit vector pointing along the 3-momentum of the photon.

- b) Show that

$$\frac{dx^i}{dt} = \frac{\hat{p}^i}{a}(1 - \Phi - \Psi)$$

to first order. Give a physical interpretation of this equation.

- c) Write down the Boltzmann equation expanded into its dynamic variables, x^i , t , p and \hat{p}^i . Why can the term dependent on \hat{p} be neglected?
- d) Starting from the 0-component of the geodesic equation, show that

$$\frac{dp}{dt} = p\frac{d\Psi}{dt} - \Gamma_{\alpha\beta}^0 \frac{P^\alpha P^\beta}{p}(1 + 2\Psi)$$

to first order.

- e) The expression derived in d) can be written out in full, and one then obtains

$$\frac{1}{p} \frac{dp}{dt} = -\frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} - H - \frac{\partial \Phi}{\partial t}. \quad (2)$$

What does this equation tell us?

- f) Using the above information, write down the Boltzmann equation for the distribution of photons. Which assumption/approximation is needed to get from this equation to an equation for $\Theta(x, t)$, where $\Theta = \frac{T(x) - T_0}{T_0}$ is the photon temperature perturbation, $T(x)$ is the physical temperature at position x , and T_0 is the average background temperature?

Problem 5 – Line-of-sight integration (AST9420)

A crucial part in speeding up a Boltzmann code is to change from direct integration of the Boltzmann-Einstein equations to a smarter approach, called “line-of-sight” integration. In this problem, we will derive the expression for the transfer function $\Theta_l(k)$ in terms of the source function.

Before we begin, let us review some relations concerning the Legendre polynomials, $P_l(\mu)$, that you may or may not find use-

ful in the following:

$$\begin{aligned}
P_0(\mu) &= 1 \\
P_1(\mu) &= \mu \\
P_l(\mu) &= (-1)^l P_l(-\mu) \\
\int_{-1}^1 P_l(\mu) P_{l'}(\mu) d\mu &= \delta_{ll'} \frac{2}{2l+1} \\
j_l(x) &= \frac{1}{2i^l} \int_{-1}^1 e^{i\mu x} P_l(\mu) d\mu \\
f_l &= \frac{i^l}{2} \int_{-1}^1 f(\mu) P_l(\mu) d\mu
\end{aligned}$$

Here $j_l(x)$ is the spherical Bessel function of order l , and $f(\mu)$ is an arbitrary function defined between -1 and 1.

Also, note that in the following, $\dot{}$ means derivative with respect to conformal time.

- a) First, from an coding point of view, why does one obtain such a large speed-up with the line-of-sight integration approach compared to direct integration?
- b) In a few sentences, explain what the main physical difference is between the line-of-sight integration and the direct solution approaches.
- c) The starting point of the line-of-sight integration method is the Boltzmann equation for photons before expanding into multipoles,

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

where $\Theta = \Theta(k, \mu, \eta)$ and μ is the angle between the photon propagation direction, \hat{p} , and the wave vector, \hat{k} . Define

$$\tilde{S} \equiv -\dot{\Phi} - ik\mu\Psi - \dot{\tau}[\Theta_0 - \Theta + \mu v_b],$$

and show that this equation can be formally solved to obtain an expression for the photon amplitude observed today given by

$$\Theta(\eta_0, k, \mu) = \int_0^{\eta_0} \tilde{S} e^{ik\mu(\eta-\eta_0)-\tau} d\eta.$$

(Note that we have dropped a quadrupole/polarization term in this expression, in order to keep things simple(r).)

- d) Assume that \tilde{S} does not depend on μ (in this sub-problem only). Show that in this case

$$\Theta_l(\eta_0, k) = (-1)^l \int_0^{\eta_0} \tilde{S} e^{-\tau} j_l[k(\eta - \eta_0)] d\eta,$$

where $\Theta_l(\eta, k)$ are the multipole expansion coefficients of $\Theta(\eta, k, \mu)$.

- e) In reality, \tilde{S} does of course depend on μ , and this have to be taken into account in the expression in c). The easiest way of doing this is by noting that \tilde{S} is multiplied with $e^{ik\mu(\eta-\eta_0)}$, and μ and $k(\eta - \eta_0)$ are therefore Fourier conjugate (just like k and x). This allows us to set

$$\mu \rightarrow \frac{1}{ik} \frac{d}{d\eta}$$

everywhere μ appears in \tilde{S} , just like we can set $ik \rightarrow d/dx$ in a standard Fourier transformation.

Use this to show that the full solution for the transfer function is

$$\Theta_l(\eta_0, k) = \int_0^{\eta_0} S(k, \eta) j_l[k(\eta_0 - \eta)] d\eta,$$

where

$$S(k, \eta) = e^{-\tau} \left[-\dot{\Phi} - \dot{\tau}\Theta_0 \right] + \frac{d}{d\eta} \left[e^{-\tau} \left(\Psi - \frac{v_b \dot{\tau}}{k} \right) \right]$$

(Hint: You may need your old knowledge about integration-by-parts to get this right :-))

- f) Introducing the visibility function, $g(x) \equiv -\dot{\tau}e^{-\tau}$, the source function can be rewritten into the form

$$S(k, \eta) = g[\Theta_0 + \Psi] + \frac{d}{d\eta} \left(\frac{gv_b}{k} \right) + e^{-\tau} \left[\dot{\Psi} - \dot{\Phi} \right].$$

What is the physical interpretation of each of these terms?

1 Appendix

1.1 General relativity

- Suppose that the structure of spacetime is described by some metric $g_{\mu\nu}$.
- The Christoffel symbols are

$$\Gamma_{\alpha\beta}^{\mu} = \frac{g^{\mu\nu}}{2} \left[\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right] \quad (3)$$

- The Ricci tensor reads

$$R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^{\alpha} - \Gamma_{\mu\alpha,\nu}^{\alpha} + \Gamma_{\beta\alpha}^{\alpha} \Gamma_{\mu\nu}^{\beta} - \Gamma_{\beta\nu}^{\alpha} \Gamma_{\mu\alpha}^{\beta} \quad (4)$$

- The Einstein equations reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu} \quad (5)$$

where $\mathcal{R} \equiv R_{\mu}^{\mu}$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

- For a perfect fluid, the energy-momentum tensor is

$$T_{\nu}^{\mu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \quad (6)$$

where ρ is the density of the fluid and p is the pressure.

1.2 Background cosmology

- Four “time” variables: $t =$ physical time, $\eta = \int_0^t a^{-1}(t)dt =$ conformal time, $a =$ scale factor, $x = \ln a$
- Friedmann-Robertson-Walker metric for flat space: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j)$
- Friedmann’s equations:

$$H \equiv \frac{1}{a} \frac{da}{dt} = H_0 \sqrt{(\Omega_m + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda} \quad (7)$$

$$\mathcal{H} \equiv \frac{1}{a} \frac{da}{d\eta} = H_0 \sqrt{(\Omega_m + \Omega_b)a^{-1} + \Omega_r a^{-2} + \Omega_\Lambda a^2} \quad (8)$$

- Conformal time as a function of scale factor:

$$\eta(a) = \int_0^a \frac{da'}{a' \mathcal{H}(a')} \quad (9)$$

1.3 The perturbation equations

Einstein-Boltzmann equations:

$$\Theta'_0 = -\frac{k}{\mathcal{H}}\Theta_1 - \Phi', \quad (10)$$

$$\Theta'_1 = -\frac{k}{3\mathcal{H}}\Theta_0 - \frac{2k}{3\mathcal{H}}\Theta_2 + \frac{k}{3\mathcal{H}}\Psi + \tau' \left[\Theta_1 + \frac{1}{3}v_b \right], \quad (11)$$

$$\Theta'_l = \frac{lk}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)k}{(2l+1)\mathcal{H}}\Theta_{l+1} + \tau' \left[\Theta_l - \frac{1}{10}\Theta_l\delta_{l,2} \right], \quad l \geq 2 \quad (12)$$

$$\Theta_{l+1} = \frac{k}{\mathcal{H}}\Theta_{l-1} - \frac{l+1}{\mathcal{H}\eta(x)}\Theta_l + \tau'\Theta_l, \quad l = l_{\max} \quad (13)$$

$$\delta' = \frac{k}{\mathcal{H}}v - 3\Phi' \quad (14)$$

$$v' = -v - \frac{k}{\mathcal{H}}\Psi \quad (15)$$

$$\delta'_b = \frac{k}{\mathcal{H}}v_b - 3\Phi' \quad (16)$$

$$v'_b = -v_b - \frac{k}{\mathcal{H}}\Psi + \tau'R(3\Theta_1 + v_b) \quad (17)$$

$$\Phi' = \Psi - \frac{k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2} [\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0] \quad (18)$$

$$\Psi = -\Phi - \frac{12H_0^2}{k^2 a^2}\Omega_r\Theta_2 \quad (19)$$

1.4 Initial conditions

$$\Phi = 1 \quad (20)$$

$$\delta = \delta_b = \frac{3}{2}\Phi \quad (21)$$

$$v = v_b = \frac{k}{2\mathcal{H}}\Phi \quad (22)$$

$$\Theta_0 = \frac{1}{2}\Phi \quad (23)$$

$$\Theta_1 = -\frac{k}{6\mathcal{H}}\Phi \quad (24)$$

$$\Theta_2 = -\frac{8k}{15\mathcal{H}\tau'}\Theta_1 \quad (25)$$

$$\Theta_l = -\frac{l}{2l+1}\frac{k}{\mathcal{H}\tau'}\Theta_{l-1} \quad (26)$$

1.5 Recombination and the visibility function

- Optical depth

$$\tau(\eta) = \int_{\eta}^{\eta_0} n_e \sigma_T a d\eta' \quad (27)$$

$$\tau' = -\frac{n_e \sigma_T a}{\mathcal{H}} \quad (28)$$

- Visibility function:

$$g(\eta) = -\dot{\tau} e^{-\tau(\eta)} = -\mathcal{H}\tau' e^{-\tau(x)} = g(x) \quad (29)$$

$$\tilde{g}(x) = -\tau' e^{-\tau} = \frac{g(x)}{\mathcal{H}}, \quad (30)$$

$$\int_0^{\eta_0} g(\eta) d\eta = \int_{-\infty}^0 \tilde{g}(x) dx = 1. \quad (31)$$

- The Saha equation,

$$\frac{X_e^2}{1 - X_e} = \frac{1}{n_b} \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b}, \quad (32)$$

where $n_b = \frac{\Omega_b \rho_c}{m_b a^3}$, $\rho_c = \frac{3H_0^2}{8\pi G}$, $T_b = T_r = T_0/a = 2.725\text{K}/a$, and $\epsilon_0 = 13.605698\text{eV}$.

- The Peebles equation,

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{n_b} \left[\beta(T_b)(1 - X_e) - n_H \alpha^{(2)}(T_b) X_e^2 \right], \quad (33)$$

where

$$C_r(T_b) = \frac{\Lambda_{2s \rightarrow 1s} + \Lambda_\alpha}{\Lambda_{2s \rightarrow 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)}, \quad (34)$$

$$\Lambda_{2s \rightarrow 1s} = 8.227\text{s}^{-1} \quad (35)$$

$$\Lambda_\alpha = H \frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \quad (36)$$

$$n_{1s} = (1 - X_e) n_H \quad (37)$$

$$\beta^{(2)}(T_b) = \beta(T_b) e^{3\epsilon_0/4T_b} \quad (38)$$

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e T_b}{2\pi} \right)^{3/2} e^{-\epsilon_0/T_b} \quad (39)$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2}{m_e^2} \sqrt{\frac{\epsilon_0}{T_b}} \phi_2(T_b) \quad (40)$$

$$\phi_2(T_b) = 0.448 \ln(\epsilon_0/T_b) \quad (41)$$

1.6 The CMB power spectrum

1. The source function:

$$\tilde{S}(k, x) = \tilde{g} \left[\Theta_0 + \Psi + \frac{1}{4}\Theta_2 \right] + e^{-\tau} [\Psi' + \Phi'] - \frac{1}{k} \frac{d}{dx} (\mathcal{H}\tilde{g}v_b) + \frac{3}{4k^2} \frac{d}{dx} \left[\mathcal{H} \frac{d}{dx} (\mathcal{H}\tilde{g}\Theta_2) \right] \quad (42)$$

$$\frac{d}{dx} \left[\mathcal{H} \frac{d}{dx} (\mathcal{H}\tilde{g}\Theta_2) \right] = \frac{d(\mathcal{H}\mathcal{H}')}{dx} \tilde{g}\Theta_2 + 3\mathcal{H}\mathcal{H}'(\tilde{g}\Theta_2 + \tilde{g}\Theta_2') + \mathcal{H}^2(\tilde{g}''\Theta_2 + 2\tilde{g}'\Theta_2' + \tilde{g}\Theta_2''), \quad (43)$$

$$\Theta_2'' = \frac{2k}{5\mathcal{H}} \left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_1 + \Theta_1' \right] + \frac{3}{10} [\tau''\Theta_2 + \tau'\Theta_2'] - \frac{3k}{5\mathcal{H}} \left[-\frac{\mathcal{H}'}{\mathcal{H}}\Theta_3 + \Theta_3' \right] \quad (44)$$

2. The transfer function:

$$\Theta_l(k, x=0) = \int_{-\infty}^0 \tilde{S}(k, x) j_l[k(\eta_0 - \eta(x))] dx \quad (45)$$

3. The CMB spectrum:

$$C_l = \int_0^\infty \left(\frac{k}{H_0} \right)^{n-1} \Theta_l^2(k) \frac{dk}{k} \quad (46)$$