# A short introduction to General Relativity

General Relativity (GR) represents our most fundamental understanding of time, space and gravity, and is absolutely necessary in order to formulate consistent cosmological models. It is a geometric theory, and can be formulated in a coordinate-free form. Unfortunately, it would take too long to do so here, and in practical calculations one anyway has to choose a coordinate system. In these lectures we therefore will use the more old-fashioned formulation of the theory. But first we need to know what tensors are.

## Tensors

Given two points $P$ and $Q$ with coordinates $x^\mu$ and $x^\mu + dx^\mu$, where $\mu = 0, 1, \ldots n-1$ (such that the space has $n$ dimensions). These two points define an infinitesmal vector $\vec{PQ}$, which we consider as 'fixed' to the starting point $P$. The components of the vector in the $x^\mu$ coordinate system are $dx^\mu$. In a different coordinate system $x'^\mu$, which is connected with the $x^\mu$ system through a transformation $x'^\mu = x'^\mu(x^\nu)$, the same vector has coordinates found by using the chain rule for differentation,

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu.$$

Here we use the convention that an index that appears twice is to be summed over, ie., in this case we sum over $\nu$. The partial derivative is to be calculated in the point $P$. A **contravariant vector** (also called a **contravariant tensor of rank 1**) is a set of quantities $A^\mu$ in the $x^\mu$ system that transforms in the same way as $dx^\mu$:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu,$$

where the partial derivatives are again to be evaluated in the point $P$. A **contravariant tensor of rank 2** is a set of $n^2$ quantities associted with the point $P$, $T^{\mu\nu}$ in the $x^\mu$ system that transform as follows:

$$T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}.$$

A simple example of such a quantity is the product of two contravariant vectors $A^\mu B^\nu$. We can define contravariant tensors of arbitrary rank by including additional $\partial x'^\mu / \partial x^\alpha$ faktors. An important special case is a contravariant tensor of rank 0, a so-called **scalar** $\phi$ which transforms as

$$\phi' = \phi$$

Let $\phi = \phi(x^\mu)$ be a scalar continuous and differentiable function, such that the derivative $\partial\phi/\partial x^\mu$ exists in all points in space. We can consider the coordinates $x^\mu$ as functions of $x'^\nu$ and write

$$\phi = \phi(x^\mu(x'^\nu)).$$

Using the chain rule to differentiate with respect to $x'^{\nu}$, we get

$$\frac{\partial \phi}{\partial x'^{\nu}} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} \frac{\partial \phi}{\partial x^{\mu}}.$$

This is the prototype of how a **covariant vector** (also called a **covariant tensor of rank 1**) transforms. This is in general a set of quantities $A_\mu$ associated with the point $x^\mu$ that transform like

$$A'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} A_{\nu}.$$

Note that $x$ and $x'$ have switched place in the partial derivative. As for contravariant tensors, we can also define covariant tensors of higher rank. For example, a covariant tensor of rank 2 transforms as

$$T'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} T_{\alpha\beta}.$$

We can also have mixed tensors. A mixed tensor of rank 3 may for instance have one contravariant and two covariant indices, and will in that case transform like

$$T'^{\mu}_{\ \nu\sigma} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\sigma}} T^{\alpha}_{\beta\gamma}.$$

Why are tensors important? For GR, this is easy to see by considering the tensor equation

$$X_{\mu\nu} = Y_{\mu\nu}.$$

This equation says that all the components of the covariant tensors $X$ and $Y$ are equal in a coordinate system $x$. However, then we also get that

$$\frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} X_{\mu\nu} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} Y_{\mu\nu}$$

and since we know that $X$ and $Y$ transform as covariant tensors of rank 2, we have that $X'_{\alpha\beta} = Y'_{\alpha\beta}$. In other words, all the coordinates of $X$ and $Y$ are equal also in the new coordinate system. That is, a tensor equation is valid in **all** coordinate systems. Tensors therefore play a critical role, because we want to define laws of nature that are valid irrespective of the reference system.

Tensors may be added and subtracted in the obvious way A tensor of rank 2 is symmetric if (shown here for covariant tensors) $X_{\mu\nu} = X_{\nu\mu}$, and anti-symmetric if $X_{\mu\nu} = -X_{\nu\mu}$. Any tensor of rank 2 may be written as a sum of a symmetric part $X_{(\mu\nu)}$ and an anti-symmetric part $X_{[\mu\nu]}$, where

$$\begin{aligned} X_{(\mu\nu)} &= \frac{1}{2}(X_{\mu\nu} + X_{\nu\mu}) \\ X_{[\mu\nu]} &= \frac{1}{2}(X_{\mu\nu} - X_{\nu\mu}) \end{aligned}$$

which you easily can check yourself. Another important operation is **contraction**: from a tensor of contravariant rank $p \geq 1$ and covariant rank $q \geq 1$ we can form a tensor of rank $(p-1, q-1)$ by putting two indices equal to each other, and sum over them For instance, we may from the tensor $X^{\mu}_{\nu\gamma\sigma}$ form another tensor $Y_{\gamma\sigma}$ by writing

$$Y_{\gamma\sigma} = X^{\mu}_{\mu\gamma\sigma}.$$

Note that if we contract a tensor of rank $(1, 1)$ we get a scalar.

## The metric tensor

A particularly important tensor $g_{\mu\nu}$ of rank 2 is the **metric tensor** or simply **the metric**. This is a symmetric tensor that defines the distance between two neighboring points $x^{\mu}$ and $x^{\mu} + dx^{\mu}$:

$$ds^2 = g_{\mu\nu}(x)dx^{\mu}dx^{\nu}.$$

It also defines the length of a vector $A$ in the point $x$,

$$A^2 = g_{\mu\nu}(x)A^{\mu}A^{\nu},$$

and the scalar product between two vectors $A$ and $B$,

$$AB = g_{\mu\nu}(x)A^{\mu}B^{\nu}.$$

Two vectors are said to be orthogonal if their scalar product is zero. In GR it is possible that the length of a vector (which is not equal to the zero vector) may be zero, $A^2 = 0$. We can view the metric as a $n \times n$ matrix, and its determinant is denoted $g$, $g = \det(g_{\mu\nu})$. If the determinant is different from zero, the metric has an inverse, and this is a contravariant tensor of rank 2 that fulfills

$$g_{\mu\alpha}g^{\alpha\nu} = \delta^{\nu}_{\mu},$$

where $\delta^{\nu}_{\mu} = 1$ for $\mu = \nu$ and zero otherwise. The tensor $g_{\mu\nu}$ can be used to raise indices, while $g^{\mu\nu}$ lowers indices. For example,

$$\begin{aligned} T^{\nu}_{\mu} &= g_{\mu\alpha}T^{\alpha\nu} \\ T^{\mu}_{\nu} &= g^{\mu\alpha}T_{\alpha\nu}. \end{aligned}$$

## The equivalence principle

General relativity starts with the equivalence principle: An observer in free fall does not feel any gravity. If he drops an object next to him, and the object initially has zero velocity, it will remain at rest even after he has dropped it. The observer therefore has the right to claim himself to be at rest. More precisely we can say that *In an arbitrary point in a gravity field we can choose a coordinate system, the so-called free-fall system, defined such that it moves with the same acceleration as an object in free fall would have had in the same point. In this system all the laws of physics will have the same form as in Special Relativity. The exception is the force of gravity, which vanishes in this system.*

Two comments are appropriate here:

1. This formulation of the equivalence principle states that the inertial masss, $m_I$ (which enters into Newton's 2. law) is equal to the gravitational mass $m_G$ (which enters into the law of gravity). If this was not the case, the observer and the object would have had different accelerations, and they would therefore not be at rest with respect to eachother.

2. Note that the formulation says *in a point*. That is, it is not possible in general to find a reference system that covers all of spacetime in which gravity is transformed away.

The equivalence principle is important because ew can use it to formulate relativistically correct equations: Start by analyzing the situation in the free-fall system, where the physics is reasonably simple. If we can formulate the result as a tensor equation, we automatically know that it will be valid in all reference systems.

## The geodesic equation

Let us use the equivalence principle to find the equation of motion for a particle in a gravity field. We start with the free-fall system, where we can use special relativity. Let the point under consideration have coordinates $\xi^\mu = (t, x, y, z)$, where we have chosen units such that $c = 1$. I follow Dodelson and write the line element as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} d\xi^\mu d\xi^\nu,$$

such that the Minkowski metric reads $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Since the particle is not affected by any forces in the free-fall system, the equation of motion is quite simply

$$\frac{d^2\xi^\mu}{d\tau^2} = 0,$$

where $d\tau^2 = -ds^2$ is **eigentime**, that is. time measured on a watch in the reference system in which the particle is at rest. This equation is a tensor equation in special relativity: Being transformed by Lorentz transformations $\xi$ is a four-vector, and $\tau$ is a scalar. But Lorentz transformations are only valid for systems that moves with constant speed with respect to each other. We therefore need an equation that invariant under more general transformations. Let us consider what happens with the above equation under a general transformation into new coordinates $x^\mu$. After such a transformation,

$$d\xi^\mu = \frac{\partial \xi^\mu}{\partial x^\nu} dx^\nu,$$

such that

$$\frac{d\xi^\mu}{d\tau} = \frac{\partial \xi^\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau}.$$

Therefore we have

$$
\begin{aligned}
0 &= \frac{d^2\xi^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial\xi^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) \\
&= \frac{\partial\xi^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{dx^\nu}{d\tau}\frac{d}{d\tau}\left(\frac{\partial\xi^\mu}{\partial x^\nu}\right) \\
&= \frac{\partial\xi^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2\xi^\mu}{\partial x^\nu\partial x^\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}
\end{aligned}
$$

Now we multiply this equation with $\partial x^\sigma/\partial\xi^\mu$ and sum over $\mu$. In the first term we then get

$$\frac{\partial x^\sigma}{\partial\xi^\mu}\frac{\partial\xi^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} = \frac{\partial x^\sigma}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} = \delta^\sigma_\nu\frac{d^2x^\nu}{d\tau^2} = \frac{d^2x^\sigma}{d\tau^2}$$

Therefore we find

$$\frac{d^2x^\sigma}{d\tau^2} + \frac{\partial x^\sigma}{\partial\xi^\mu}\frac{\partial^2\xi^\mu}{\partial x^\nu\partial x^\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0.$$

We write this equation as

$$\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0,$$

where

$$\Gamma^\sigma_{\nu\rho} = \frac{\partial x^\sigma}{\partial\xi^\mu}\frac{\partial^2\xi^\mu}{\partial x^\nu\partial x^\rho}$$

is called the **Christoffel symbol** or the **connection**. In the new coordinates the expression for the eigentime is given by

$$d\tau^2 = -\eta_{\mu\nu}d\xi^\mu d\xi^\nu = -\eta_{\mu\nu}\frac{\partial\xi^\mu}{\partial x^\alpha}\frac{\partial\xi^\nu}{\partial x^\beta}dx^\alpha dx^\beta \equiv -g_{\alpha\beta}dx^\alpha dx^\beta,$$

whereas the metric in the new coordinates is

$$g_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial\xi^\mu}{\partial x^\alpha}\frac{\partial\xi^\nu}{\partial x^\beta}.$$

You can now convince yourself that the new equation of motion, which is called the **geodesic equation**, remains on the same form if we apply a general coordinate transformation. What you need to know is that the Christoffel symbol is *not* a tensor, but transforms as

$$\Gamma'^\alpha_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\delta}\frac{\partial x^\eta}{\partial x'^\beta}\frac{\partial x^\phi}{\partial x'^\gamma}\Gamma^\delta_{\eta\phi} + \frac{\partial x'^\alpha}{\partial x^\delta}\frac{\partial^2 x^\delta}{\partial x'^\beta\partial x'^\gamma}.$$

If we introduce the covariant derivative

$$\nabla_\gamma A^\alpha = \frac{\partial A^\alpha}{\partial x^\gamma} + \Gamma^\alpha_{\beta\gamma}A^\beta,$$

(often written as $A^\alpha_{;\gamma}$) we can show that $\nabla_\gamma A^\alpha$ transforms as a mixed tensor of rank 2. The trajectory of the particle is given by $x^\mu(\tau)$. The tangent vector to the trajectory in a given point is given by $X^\mu = dx^\mu/d\tau$, which transforms as a contravariant vector. It is then straightforward to show (do it!) that the geodesic equation may be written on the form

$$X^\gamma \nabla_\gamma X^\alpha = 0.$$

Often we use the notation $\nabla_\gamma X^\alpha = X^\alpha_{;\gamma}$, and $\frac{\partial X^\alpha}{\partial x^\gamma} = X^\alpha_{,\gamma}$. In that case we can write the geodesic equation simply as

$$X^\gamma X^\alpha_{;\gamma} = 0.$$

Written on this form, it is obvious that the geodesic equation is a tensor equation, and therefore it is valid in all reference systems. Note that the equation is invariant under general transformations, also pure coordinate transformations, such as switching from cartesian to spherical coordinates. This implies that it can some times be difficult to distinguish physical effects from coordinate effects in general relativity. This is a problem that emerges in cosmological perturbation theory, and is called the **gauge problem**.

In hindsight we can also 'derive' the geodesic equation in a very simple manner. In the free-fall system the tangent vector to the particle trajectory is given by $\Xi^\mu = d\xi^\mu/d\tau$, and the equation of motion may be written as

$$\frac{d}{d\tau}\left(\frac{d\xi^\mu}{d\tau}\right) = \frac{d}{d\tau}\Xi^\mu = 0.$$

Using the chain rule, we can rewrite this into

$$\frac{d\xi^\nu}{d\tau}\frac{\partial \Xi^\mu}{\partial \xi^\nu} = \Xi^\nu \Xi^\mu_{,\nu} = 0.$$

Partial differentation is not a tensor operation, but covariant derivation is. In the free-fall system these two operations are the same (since all the Christoffel symbols are zero in this system), and we can therefore replace the comma with a semi-colon:

$$\Xi^\nu \Xi^\mu_{;\nu} = 0.$$

Now we have translated the equation of motion into the form of a tensor equation, and therefore it is valid in all reference systems.

Note that $\Gamma$ and $g_{\mu\nu}$ are geometric quantities. Gravity is integrated into the geometry of spacetime, and has therefore become a geometric effect. Since both $\Gamma$ and $g$ are geometric quantities, it may not come as a surprise that there is a connection between them. I write this connection without proof:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right).$$

The metric is therefore an extremely important object in GR: If we know this, we also know the geometry of spacetime, and the geometry of spacetime determines how particles move.

A little more about the covariant derivative at the end. In general for a mixed tensor, we have that the covariant derivative is given by

$$\nabla_\gamma T^{\alpha\cdots}_{\beta\cdots} = \frac{\partial}{\partial x^\gamma} T^{\alpha\cdots}_{\beta\cdots} + \Gamma^\alpha_{\delta\gamma} T^{\delta\cdots}_{\beta\cdots} + \cdots - \Gamma^\delta_{\beta\gamma} T^{\alpha\cdots}_{\delta\cdots} - \cdots,$$

such that each contravariant index gives rise to a Christoffel symbol with a positive sign, while each covariant index gives rise to a Christoffel symbol with negative sign. For a contravariant tensor of rank two, the expression reads

$$\nabla_\gamma T^{\mu\nu} = \frac{\partial}{\partial x^\gamma} T^{\mu\nu} + \Gamma^\mu_{\beta\gamma} T^{\beta\nu} + \Gamma^\nu_{\beta\gamma} T^{\mu\beta}.$$

## Newtonian limit

We consider a particle that moves slowly in a weak static gravity field. Remember that we have set $c = 1$, such that slow motion implies

$$\frac{dx^i}{dt} \ll 1.$$

For the geodesic equation this means that all terms that contain $dx^i/d\tau$ ($i = 1, 2, 3$) can be neglected compared to the $(dt/d\tau)^2$ term, such that we get

$$\frac{d^2 x^\sigma}{d\tau^2} + \Gamma^\sigma_{00} \left( \frac{dt}{d\tau} \right)^2 = 0.$$

That the gravity field is weak means that the metric cannot be too different from the Minkowski metric for flat spacetime. We therefore write it as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $|h_{\mu\nu}| \ll 1$. We can therefore neglect all terms that include more than one $h_{\mu\nu}$ factor. That the gravity field is static means that

$$\frac{\partial g_{\mu\nu}}{\partial t} = \frac{\partial h_{\mu\nu}}{\partial t} = 0.$$

The Christoffel symbol we need is then

$$\Gamma^\sigma_{00} = \frac{g^{\rho\sigma}}{2} \left( \frac{\partial g_{0\rho}}{\partial x^0} + \frac{\partial g_{0\rho}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\rho} \right) = -\frac{1}{2} \eta^{\rho\sigma} \frac{\partial h_{00}}{\partial x^\rho}.$$

For $\sigma = i = 1, 2, 3$ the geodesic equation becomes

$$\frac{d^2 x^i}{d\tau^2} = \frac{\eta^{i\rho}}{2} \frac{\partial h_{00}}{\partial x^\rho} \left( \frac{dt}{d\tau} \right)^2 = \frac{1}{2} \left( \frac{dt}{d\tau} \right)^2 \frac{\partial h_{00}}{\partial x^i}.$$

Further, for $\sigma = 0$ we see that $\Gamma^0_{00} \propto \frac{\partial h_{00}}{\partial t} = 0$, such that this component of the geodesic equations simply becomes

$$\frac{d^2 t}{d\tau^2} = 0,$$

such that $dt/d\tau$ = konstant. Then we can divide the equation for $\sigma = i$ with $(dt/d\tau)^2$ and obtain

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}.$$

For a particle in a static gravity potential $\Psi$ in Newtonian mechanics the equation of motion would be

$$\frac{d^2x^i}{dt^2} = -\frac{\partial \Psi}{\partial x^i},$$

and we therefore see that if we identify $h_{00} = -2\Psi$, then the geodesic equation gives us the Newtonian limit. In other words, the 00 component of the metric must be $g_{00} = -1 - 2\Psi$.

## Important tensors in GR

As mentioned, the Christoffel symbol is not a tensor. But it can be used to construct tensors related to the curvature of spacetime. What we need to know is the **Riemanntensor**

$$R^{\mu}_{\sigma\beta\alpha} = \Gamma^{\mu}_{\sigma\alpha,\beta} - \Gamma^{\mu}_{\sigma\beta,\alpha} + \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\sigma\alpha} - \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\sigma\beta},$$

where I have introduced the notation

$$_{,\alpha} = \frac{\partial}{\partial x^{\alpha}}.$$

**The Ricci tensor** is found by contracting two indices in the Riemann tensor:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}.$$

**The Ricci scalar** is given by

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu}.$$

Finally, the **Einstein tensor** is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R}.$$

The Einstein tensor has the important property that its covariant divergens is equal to zero: $G^{\mu\nu}_{;\nu} = 0$.

## The energy-momentum tensor

Let us return to the special theory of relativity for a moment. We consider a system consisting of non-interacting particules with density described by the function $\rho(x)$. This function represents the density measured by an observer moving with the particle flow, characterized by the four-velocity field

$$u^{\mu} = \frac{dx^{\mu}}{d\tau}.$$

From these equations we can form a contravariant tensor of rank 2:

$$T^{\mu\nu} = \rho u^\mu u^\nu.$$

In special relativity we have

$$u^\mu = \gamma(1, \vec{u}),$$

where $\vec{u} = d\vec{x}/dt$,

$$d\tau^2 = -ds^2 = dt^2 - d\vec{x}^2 = dt^2(1 - u^2) = \frac{dt^2}{\gamma^2},$$

and $\gamma = (1 - u^2)^{-1/2}$. If we write $T$ as a matrix, it looks like this:

$$(T^{\mu\nu}) = \rho \begin{pmatrix} 1 & u_x & u_y & u_z \\ u_x & u_x^2 & u_x u_y & u_x u_z \\ u_y & u_x u_y & u_y^2 & u_y u_z \\ u_z & u_x u_z & u_y u_z & u_z^2 \end{pmatrix}$$

From fluid dynamics we know the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0,$$

which expresses conservation of mass. With our definition of $T$ we can write this equation on a very compact form: Computing $T^{0\nu}_{,\nu}$, we find

$$T^{0\nu}_{,\nu} = \frac{\partial T^{0\nu}}{\partial x^\nu} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}).$$

Therefore the continuity equation may be written as

$$T^{0\nu}_{,\nu} = 0.$$

Another important equation in hydrodynamics, the Navier-Stokes equation, says that for a fluid without internal pressure and external forces, we have

$$\rho \left[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right] = 0.$$

Physically, this equation says that momentum is conserved. One can show that this equation on component form, with our choice of $T$, may be written as

$$T^{i\nu}_{,\nu} = 0,$$

for $i = 1, 2, 3$. We can therefore combine conservation of mass and momentum for our system in a very elegant manner through the equation

$$T^{\mu\nu}_{,\nu} = 0.$$

Written on this form, we also immediately see that the generalization of this equation to GR should be

$$T^{\mu\nu}_{;\nu} = 0.$$

Becuase this summarize conservation of mass (energy) and momentum it is called the **energy-momentum tensoren**.

We will deal mostly with the case of a **perfect fluid**. This is characterized by the energy-density , $\rho = \rho(x)$, the pressure $p = p(x)$ and the four-velocity $u^\mu = dx^\mu/d\tau$. If the pressure $p = 0$, we should end up with the same energy-momentum tensor as in last section. That implies that we should choose

$$T^{\mu\nu} = \rho u^\mu u^\nu + pS^{\mu\nu},$$

where $S^{\mu\nu}$ is a symmetric tensor. The only symmetric tensors of rank 2 that are connected with the fluid are $u^\mu u^\nu$ and the metric tensor $g^{\mu\nu}$. The simplest choice is therefore

$$S^{\mu\nu} = Au^\mu u^\nu + Bg^{\mu\nu},$$

where $A$ and $B$ are constants. We take advantage of the fact that the tensor should have a special-relativistic limit that we can live with. In this limit we have $g^{\mu\nu} = \eta^{\mu\nu}$, the Minkowski metric. What we want most in the entire world is that $T^{\mu\nu}_{,\nu}$ once again shall express conservation of energy and momentum. If we demand that $T^{0\nu}_{,\nu} = 0$ should give us the continuity equation, and $T^{i\nu}_{,\nu} = 0$ should give us Navier-Stokes (this time with pressure terms, but without external forces), we find that this is only satisfied if $A = B = 1$. We therefore choose

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu},$$

which does satisfy $T^{\mu\nu}_{;\nu} = 0$: We say that it has covariant divergence equal to 0.

## Einstein's field equations

We now have two tensors, $G_{\mu\nu}$ and $T_{\mu\nu}$, both having covariant divergence equal to zero. One, the Einstein tensor, has to do with geometry. The other, the energy-momentum tensor, has to do with mass, energy and pressure. Einstein postulated that these two tensors are proportional,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where the factor $8\pi G$ is included in order to get the right Newtonian limit. These are Einstein's field equations, one of the highlights in the intellectual history of mankind (and I say this without any trace of irony). Note that Einstein's field equations cannot be derived mathematically from the postulates of relativity, but the above choice is the simplest choice that are consistent with them. If so desired, one can formulate far more complicated equations. The right form must be decided by nature. So far it has not given us any important reasons to believe that Einstein's choice was wrong.

## The Friedmann equations

For a given energy-momentum tensor it can be very complicated to solve the Einstein equations. However, in some cases $T_{\mu\nu}$ has symmetries that simplifies the problem to the extent that we can do it analytically. One such case is for a homogeneous, isotropic universe filled with one or more perfect fluids. Such a universe is described by the Robertson-Walker metric. In this course we will limit ourselves to flat universe models, and the metric then reads

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2),$$

in spherical coordinates. For us it is easier to work with Cartesian coordinates, and then it simply reads

$$ds^2 = -dt^2 + a^2(t)((dx^1)^2 + (dx^2)^2 + (dx^3)^2) = -dt^2 + a^2(t)d\vec{x}^2.$$

Recall that $x^i$ is defined in co-moving coordinates, ie., they are stuck to objects that follows the expansion, and are constant in time. Physical distances are found by multiplying with the scale factor $a(t)\vec{x}$.

We can now write down the Einstein equations for the universe. We first need the Christoffel symbols. These are given by

$$\Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}\right).$$

From the Robertson-Walker metric we find that

$$g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2),$$

and therefore

$$g^{\mu\nu} = \mathrm{diag}(-1, 1/a^2, 1/a^2, 1/a^2).$$

For $\mu = 0$ we find

$$\begin{aligned}
\Gamma^0_{\alpha\beta} &= \frac{g^{0\nu}}{2}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \\
&= -\frac{1}{2}(g_{\alpha 0,\beta} + g_{\beta 0,\alpha} - g_{\alpha\beta,0}) \\
&= +\frac{1}{2}g_{\alpha\beta,0},
\end{aligned}$$

which gives $\Gamma^0_{00} = 0$, $\Gamma^0_{0i} = \Gamma^0_{i0} = 0$, and

$$\Gamma^0_{ij} = \frac{1}{2}\delta_{ij}\frac{\partial}{\partial t}a^2 = a\dot{a}\delta_{ij}.$$

For $\mu = i$ we find

$$\begin{aligned}
\Gamma^i_{\alpha\beta} &= \frac{1}{2}g^{i\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}) \\
&= \frac{1}{2a^2}(g_{\alpha i,\beta} + g_{\beta i,\alpha} - g_{\alpha\beta,i}) \\
&= \frac{1}{2a^2}(g_{\alpha i,\beta} + g_{\beta i,\alpha}),
\end{aligned}$$

such that

$$\Gamma^i_{00} = \frac{1}{2a^2}(g_{0i,0} + g_{0i,0}) = 0,$$

$$\Gamma^i_{0j} = \frac{1}{2a^2}(g_{0i,j} + g_{ji,0}) = \frac{1}{2a^2}\delta_{ij}\frac{\partial}{\partial t}a^2 = \frac{\dot{a}}{a}\delta_{ij} = \Gamma^i_{j0},$$

and

$$\Gamma^i_{jk} = \frac{1}{2a^2}(g_{ji,k} + g_{ki,j}) = 0.$$

The only non-zero Christoffel symbols are therefore

$$\begin{aligned} \Gamma^0_{ij} &= \delta_{ij} a\dot{a} \\ \Gamma^i_{0j} &= \Gamma^i_{j0} = \delta_{ij}\frac{\dot{a}}{a}. \end{aligned}$$

Now we are ready to compute the Ricci tensor

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}.$$

We find that

$$\begin{aligned} R_{00} &= \Gamma^\alpha_{00,\alpha} - \Gamma^\alpha_{0\alpha,0} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} \\ &= -\Gamma^i_{0i,0} - \Gamma^i_{j0}\Gamma^j_{0i} \\ &= -\sum_i \delta_{ii}\frac{\partial}{\partial t}\frac{\dot{a}}{a} - \sum_{i,j}\delta_{ij}\frac{\dot{a}}{a}\delta_{ij}\frac{\dot{a}}{a} \\ &= -3\frac{\ddot{a}a - \dot{a}^2}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 \sum_i \sum_j \delta_{ij}\delta_{ij} \\ &= -3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 - 3\left(\frac{\dot{a}}{a}\right)^2 \\ &= -3\frac{\ddot{a}}{a}. \end{aligned}$$

In some of these calculations I have written out the summations in a desparate attempt to make the calculations a little more transparent. We have to keep computing:

$$\begin{aligned} R_{0i} &= \Gamma^\alpha_{0i,\alpha} - \Gamma^\alpha_{0\alpha,i} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{0i} - \Gamma^\alpha_{\beta i}\Gamma^\beta_{0\alpha} \\ &= \Gamma^0_{\beta 0}\Gamma^\beta_{0i} + \Gamma^j_{\beta j}\Gamma^\beta_{0i} - \Gamma^0_{\beta i}\Gamma^\beta_{0i} - \Gamma^j_{\beta i}\Gamma^\beta_{0j} = 0. \end{aligned}$$

And now a real bad-boy:

$$\begin{aligned} R_{ij} &= \Gamma^\alpha_{ij,\alpha} - \Gamma^\alpha_{i\alpha,j} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{ij} - \Gamma^\alpha_{\beta j}\Gamma^\beta_{i\alpha} \\ &= \Gamma^0_{ij,0} - 0 + \Gamma^\alpha_{0\alpha}\Gamma^0_{ij} - \Gamma^0_{\beta j}\Gamma^\beta_{i0} - \Gamma^k_{\beta j}\Gamma^\beta_{ik} \\ &= \frac{\partial}{\partial t}(\delta_{ij}a\dot{a}) + \delta_{ij}a\dot{a}\sum_k \delta_{kk}\frac{\dot{a}}{a} - \Gamma^0_{kj}\Gamma^k_{i0} - \Gamma^k_{0j}\Gamma^0_{ik} \end{aligned}$$

$$
\begin{aligned}
&= \delta_{ij}(\dot{a}^2 + a\ddot{a}) + 3\delta_{ij}\dot{a}^2 - \sum_k \delta_{kj} a\dot{a}\delta_{ki}\frac{\dot{a}}{a} - \sum_k \delta_{kj}\frac{\dot{a}}{a}\delta_{ik}a\dot{a} \\
&= \delta_{ij}(\dot{a}^2 + a\ddot{a}) + 3\delta_{ij}\dot{a}^2 - \delta_{ij}\dot{a}^2 - \delta_{ij}\dot{a}^2 \\
&= \delta_{ij}(2\dot{a}^2 + a\ddot{a}).
\end{aligned}
$$

Compared to this, the Ricci scalar is a piece of cake:

$$
\begin{aligned}
\mathcal{R} &= g^{\mu\nu}R_{\mu\nu} = -R_{00} + \frac{1}{a^2}\sum_i R_{ii} \\
&= 3\frac{\ddot{a}}{a} + \frac{1}{a^2}\sum_i \delta_{ii}(2\dot{a}^2 + a\ddot{a}) \\
&= 3\frac{\ddot{a}}{a} + \frac{3}{a^2}(2\dot{a}^2 + a\ddot{a}) \\
&= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right].
\end{aligned}
$$

Vi kan endelig finne f.eks. 00-komponenten av Einsteintensoren:

$$
\begin{aligned}
G_{00} &= R_{00} - \frac{1}{2}g_{00}\mathcal{R} \\
&= -3\frac{\ddot{a}}{a} + \frac{1}{2}6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \\
&= 3\left(\frac{\dot{a}}{a}\right)^2.
\end{aligned}
$$

To find the 00-component of the Einstein equation, we also need the corresponding component of the energy-momentum tensor. For a perfect fluid, this is given by

$$
T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}.
$$

In co-moving coordinates the fluid has a three-velocity equal to 0, such that $u^\mu = (1, \vec{0})$, and

$$
T^{\mu\nu} = \begin{pmatrix} \rho + p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & p/a^2 & 0 & 0 \\ 0 & 0 & p/a^2 & 0 \\ 0 & 0 & 0 & p/a^2 \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p/a^2 & 0 & 0 \\ 0 & 0 & p/a^2 & 0 \\ 0 & 0 & 0 & p/a^2 \end{pmatrix}.
$$

Then we get

$$
T_{00} = g_{0\alpha}g_{0\beta}T^{\alpha\beta} = (-1)(-1)T^{00} = T^{00} = \rho,
$$

and the 00-component of the Einstein equation becomes,

$$
G_{00} = 8\pi G T_{00},
$$

which therefore gives

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho,$$

or

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho.$$

The attentive reader will recognize this equation as the Friedmann equation for $\dot{a}$ in the case of $k = 0$ (spatially flat universe).

We recall that the covariant divergence of the energy-momentum tensor is equal to zero:

$$T^{\mu\nu}_{;\nu} = 0.$$

If we consider the $\mu = 0$ component of this equation, we get a result you may recognize:

$$\begin{aligned}
T^{0\nu}_{;\nu} &= T^{0\nu}_{,\nu} + \Gamma^0_{\beta\nu}T^{\beta\nu} + \Gamma^\nu_{\beta\nu}T^{0\beta} \\
&= T^{00}_{,0} + \Gamma^0_{\beta\nu}T^{\beta\nu} + \Gamma^\nu_{0\nu}T^{00} \\
&= \frac{\partial\rho}{\partial t} + \Gamma^0_{ij}T^{ij} + \Gamma^i_{0i}\rho \\
&= \frac{\partial\rho}{\partial t} + \sum_i\sum_j \delta_{ij}a\dot{a}\delta_{ij}\frac{p}{a^2} + \sum_i \frac{\dot{a}}{a}\rho \\
&= \frac{\partial\rho}{\partial t} + 3\frac{\dot{a}}{a}(\rho + p),
\end{aligned}$$

where I have used the earlier expressions for the Christoffel symboles for the RW metric, and the expression for the energy-momentum tensor. Therefore, the $\mu = 0$ component of $T^{\mu\mu}_{;\nu} = 0$ gives that

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0,$$

which many may recognize from AST4220, where we derived this equation in a much simpler, but more dodgy, way.