Numerical project
Milestone III

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The evolution of perturbations

- We have derived the full set of Einstein-Boltzmann equations for how perturbations of the photons, baryons, CDM and the metric evolve.

- These depends on
  1. The background evolution (we have this from Milestone I)
  2. The recombination history (we have this from Milestone II)

- We know how to solve coupled ODEs and have what we need to be able to integrate these perturbations from the early Universe until today so we are all set. Doing this job is the main objective of Milestone III.

- This will then again be used in Milestone IV to finally compute directly observable quantities like the CMB angular power-spectrum and the matter power-spectrum

The Einstein-Boltzmann System

Photon temperature multipoles:

$$\Theta_0' = -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi',$$

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right]$$

$$\Theta_\ell' = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell-1} - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell+1} + \tau'\left[\Theta_\ell - \frac{1}{10}\Pi\delta_{\ell,2}\right], \qquad 2 \leq \ell < \ell_{\rm max}$$

$$\Theta_\ell' = \frac{ck}{\mathcal{H}}\Theta_{\ell-1} - c\frac{\ell+1}{\mathcal{H}\eta(x)}\Theta_\ell + \tau'\Theta_\ell, \qquad \ell = \ell_{\rm max}$$

‘Pressure force’ (Photon-Electron scattering)

Mass/Energy conservation

Cold dark matter and baryons:

Free streaming

Gravitational Force

$$\delta_{\rm CDM}' = \frac{ck}{\mathcal{H}}v_{\rm CDM} - 3\Phi'$$

$$v_{\rm CDM}' = -v_{\rm CDM} - \frac{ck}{\mathcal{H}}\Psi$$

$$\delta_b' = \frac{ck}{\mathcal{H}}v_b - 3\Phi'$$

$$v_b' = -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b)$$

Neutrinos and Polarization

$$\Theta_{P0}' = -\frac{ck}{\mathcal{H}}\Theta_{P1} + \tau'\left[\Theta_{P0} - \frac{1}{2}\Pi\right]$$

$$\Theta_{P\ell}' = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\Theta^P_{\ell-1} - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\Theta^P_{\ell+1} + \tau'\left[\Theta^P_\ell - \frac{1}{10}\Pi\delta_{\ell,2}\right], \qquad 1 \leq \ell < \ell_{\rm max}$$

$$\Theta_{P,\ell}' = \frac{ck}{\mathcal{H}}\Theta^P_{\ell-1} - c\frac{\ell+1}{\mathcal{H}\eta(x)}\Theta^P_\ell + \tau'\Theta^P_\ell, \qquad \ell = \ell_{\rm max}$$

$$\mathcal{N}_0' = -\frac{ck}{\mathcal{H}}\mathcal{N}_1 - \Phi',$$

$$\mathcal{N}_1' = \frac{ck}{3\mathcal{H}}\mathcal{N}_0 - \frac{2ck}{3\mathcal{H}}\mathcal{N}_2 + \frac{ck}{3\mathcal{H}}\Psi$$

$$\mathcal{N}_\ell' = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\mathcal{N}_{\ell-1} - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\mathcal{N}_{\ell+1}, \qquad 2 \leq \ell < \ell_{\rm max,\nu}$$

$$\mathcal{N}_\ell' = \frac{ck}{\mathcal{H}}\mathcal{N}_{\ell-1} - c\frac{\ell+1}{\mathcal{H}\eta(x)}\mathcal{N}_\ell, \qquad \ell = \ell_{\rm max,\nu}$$

Metric perturbations:

Poisson equation

$$\Phi' = \Psi - \frac{c^2k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_{\rm CDM}a^{-1}\delta_{\rm CDM} + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0 + 4\Omega_\nu a^{-2}\mathcal{N}_0\right]$$

$$\Psi = -\Phi - \frac{12H_0^2}{c^2k^2a^2}\left[\Omega_r\Theta_2 + \Omega_\nu\mathcal{N}_2\right]$$

Anisotropic stress

A complicated mess of equations, but containing a lot of nice physics from which we can understand qualitatively how the solution will be!

The Einstein-Boltzmann System

Note that every term that multiplies the perturbations are functions of time and scale that we know from the first two milestones

Photon temperature multipoles:

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$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right],$$

$$\Theta_\ell' = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell-1} - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell+1} + \tau'\left[\Theta_\ell - \frac{1}{10}\Pi\delta_{\ell,2}\right], \qquad 2 \le \ell < \ell_{\max}$$

$$\Theta_\ell' = \frac{ck}{\mathcal{H}}\Theta_{\ell-1} - c\frac{\ell+1}{\mathcal{H}\eta(x)}\Theta_\ell + \tau'\Theta_\ell, \qquad \ell = \ell_{\max}$$

Cold dark matter and baryons:

$$\delta_{\mathrm{CDM}}' = \frac{ck}{\mathcal{H}}v_{\mathrm{CDM}} - 3\Phi'$$

$$v_{\mathrm{CDM}}' = -v_{\mathrm{CDM}} - \frac{ck}{\mathcal{H}}\Psi$$

$$\delta_b' = \frac{ck}{\mathcal{H}}v_b - 3\Phi'$$

$$v_b' = -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b)$$

Neutrinos + Polarization

$$\Theta_{P0}' = -\frac{ck}{\mathcal{H}}\Theta_{P1} + \tau'\left[\Theta_{P0} - \frac{1}{2}\Pi\right]$$

$$\Theta_{P\ell}' = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell-1}^P - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell+1}^P + \tau'\left[\Theta_\ell^P - \frac{1}{10}\Pi\delta_{\ell,2}\right], \qquad 1 \le \ell < \ell_{\max}$$

$$\Theta_{P,\ell}' = \frac{ck}{\mathcal{H}}\Theta_{\ell-1}^P - c\frac{\ell+1}{\mathcal{H}\eta(x)}\Theta_\ell^P + \tau'\Theta_\ell^P, \qquad \ell = \ell_{\max}$$

$$\mathcal{N}_0' = -\frac{ck}{\mathcal{H}}\mathcal{N}_1 - \Phi',$$

$$\mathcal{N}_1' = \frac{ck}{3\mathcal{H}}\mathcal{N}_0 - \frac{2ck}{3\mathcal{H}}\mathcal{N}_2 + \frac{ck}{3\mathcal{H}}\Psi$$

$$\mathcal{N}_\ell' = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\mathcal{N}_{\ell-1} - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\mathcal{N}_{\ell+1}, \qquad 2 \le \ell < \ell_{\max,\nu}$$

$$\mathcal{N}_\ell' = \frac{ck}{\mathcal{H}}\mathcal{N}_{\ell-1} - c\frac{\ell+1}{\mathcal{H}\eta(x)}\mathcal{N}_\ell, \qquad \ell = \ell_{\max,\nu}$$

Metric perturbations:

$$\Phi' = \Psi - \frac{c^2k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_{\mathrm{CDM}}a^{-1}\delta_{\mathrm{CDM}} + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0 + 4\Omega_\nu a^{-2}\mathcal{N}_0\right]$$

$$\Psi = -\Phi - \frac{12H_0^2}{c^2k^2a^2}\left[\Omega_r\Theta_2 + \Omega_\nu\mathcal{N}_2\right]$$

(Note: Psi does not need to be solved for - follows from other perturbations)

Put together this is a closed set of coupled first order equations
We can solve this with an ODE solver!

The ODE system

The system can be written on the form

$$\frac{d\vec{Y}}{dx} = \vec{A}(\vec{y})$$

Where A is the “right hand side” that you need to implement

Each Y[i] represent one component and you need to specify what Y[i] means in your code and implement the equations that is consistent with this!

For example if we choose that:

$$Y[0] = \delta_{\rm CDM}$$
$$Y[1] = v_{\rm CDM}$$
$$Y[2] = \delta_{\rm b}$$
$$Y[3] = v_{\rm b}$$
$$Y[4] = \Phi$$
$$Y[5] = \Theta_0$$
$$Y[6] = \dots$$

Then the equation:

$$\delta_{\rm CDM}' = \frac{ck}{\mathcal{H}} v_{\rm CDM} - 3\Phi'$$

Would become:

$$\frac{dY[0]}{dx} = \frac{ck}{\mathcal{H}} Y[1] - 3\frac{dY[4]}{dx}$$

Filling the vector dydx[i] is what you do in the “set right hand side” routines

(NB: for this particular one if you were to write it as above dydx[4] would have to be set first otherwise it would lead to nonsense)

One small problem: Tight coupling

- Unfortunately the equation system is numerically unstable in the early Universe due to the tight coupling between photons and baryons

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right]$$

Huge

$$v_b' = -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau' R(3\Theta_1 + v_b)$$

Tiny

- The high efficiency of Compton scattering early on washes out all higher moments and only the first 2 moments (0 = ‘Overdensity’ and 1 = ‘Velocity’) are really relevant. Forces the photon velocity to be equal to the baryon velocity: they move as one **single** fluid

- Tiny * **Huge** leads to problems! Small errors in the velocities becomes big errors in the equation. We must find a way of avoiding this

One small problem: Tight coupling

- **Solution:** in the tight coupling regime we only include the first two photon multipoles 0 and 1 (and no polarisation multipoles, but all the rest of the perturbations) and use a more stable approximation for the ODE (see the website for a derivation)

$$q = \frac{-[(1-R)\tau' + (1+R)\tau''](3\Theta_1 + v_b) - \frac{ck}{\mathcal{H}}\Psi + (1 - \frac{\mathcal{H}'}{\mathcal{H}})\frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Theta_0'}{(1+R)\tau' + \frac{\mathcal{H}'}{\mathcal{H}} - 1}$$

$$v_b' = \frac{1}{1+R}\left[-v_b - \frac{ck}{\mathcal{H}}\Psi + R(q + \frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Psi)\right]$$

$$\Theta_1' = \frac{1}{3}(q - v_b')$$

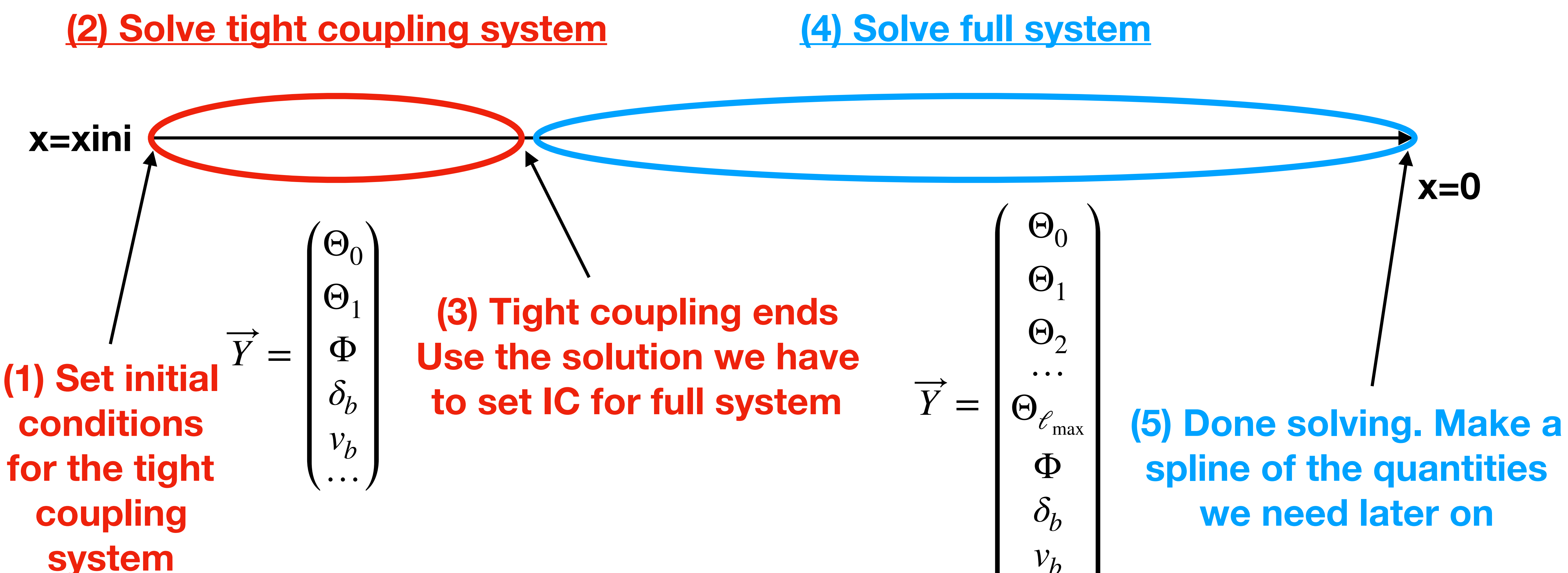
- Just a bit more complicated expressions, but not that bad (Here we also see why we needed to compute the derivatives of tau)

- Tight coupling is valid as long as we are before recombination and that

$$\left|\frac{d\tau}{dx}\right| < 10 \cdot \min(1, \frac{ck}{\mathcal{H}})$$

Summary of what to do

For every single wavenumber (and we need about 100 wave-numbers) we need to:



$$\vec{Y} = \begin{pmatrix} \Theta_0 \\ \Theta_1 \\ \Phi \\ \delta_b \\ v_b \\ \dots \end{pmatrix}$$

$$\vec{Y} = \begin{pmatrix} \Theta_0 \\ \Theta_1 \\ \Theta_2 \\ \dots \\ \Theta_{\ell_{\max}} \\ \Phi \\ \delta_b \\ v_b \\ \dots \end{pmatrix}$$

NB: what Y is in tight coupling is different from what Y is in the full system. You need to keep track of this!

The Perturbation class

```
class Perturbations{
  private:

    BackgroundCosmology *cosmo = nullptr;
    RecombinationHistory *rec  = nullptr;

    // The scales we integrate over
    const int n_k        = 100;
    const double k_min   = Constants.k_min;
    const double k_max   = Constants.k_max;

    // Start and end of the time-integration
    const int n_x        = 1000;
    const double x_start = Constants.x_start;
    const double x_end   = Constants.x_end;

    // Below is a full list of splines you probably need,
    // but you only need to make the splines you will need

    // Splines of scalar perturbations quantities
    Spline2D delta_cdm_spline{"delta_cdm_spline"};
    Spline2D delta_b_spline{"delta_b_spline"};
    Spline2D v_cdm_spline{"v_cdm_spline"};
    Spline2D v_b_spline{"v_b_spline"};
    Spline2D Phi_spline{"Phi_spline"};
    Spline2D Pi_spline{"Pi_spline"};
    Spline2D Psi_spline{"Psi_spline"};

    // Splines of source functions (ST for temperature; SE for polarization)
    Spline2D ST_spline{"ST_spline"};
    Spline2D SE_spline{"SE_spline"};

    // Splines of mulipole quantities
    std::vector<Spline2D> Theta_spline;
    std::vector<Spline2D> Theta_p_spline;
    std::vector<Spline2D> Nu_spline;

    //==========================================================
    // [1] Tight coupling ODE system
    //==========================================================

    // Set the initial conditions at the start (which is in tight coupling)
    Vector set_ic(
        const double x,
        const double k) const;

    // Right hand side of the ODE in the tight coupling regime
    int rhs_tight_coupling_ode(double x, double k, const double *y, double *dydx);

    // Compute the time when tight coupling ends
    double get_tight_coupling_time(const double k) const;
```

The objects made in Milestone 1 and 2

Number of k values and the k-range to integrate for

The x-range to integrate over (e.g. -12 -> 0.0)

Splines of the perturbations to make after done solving

Function to set IC in the start

Right hand side of The tight coupling system

```
    //==========================================================
    // [2] The full ODE system
    //==========================================================

    // Set initial condition after tight coupling
    Vector set_ic_after_tight_coupling(
        const Vector &y_tight_coupling,
        const double x,
        const double k) const;

    // Right hand side of the ODE in the full regime
    int rhs_full_ode(double x, double k, const double *y, double *dydx);

    //==========================================================
    // [3] Integrate the full system
    //==========================================================

    // Integrate perturbations and spline the result
    void integrate_perturbations();

    //==========================================================
    // [4] Compute source functions from the result
    //==========================================================

    // Compute source functions and spline the result
    void compute_source_functions();

  public:

    // Constructors
    Perturbations() = default;
    Perturbations(
        BackgroundCosmology *cosmo,
        RecombinationHistory *rec);

    // Do all the solving
    void solve();

    // Print some useful info about the class
    void info() const;

    // Output info to file
    void output(const double k, const std::string filename) const;

    // Get the quantities we have integrated
    double get_delta_cdm(const double x, const double k) const;
    double get_delta_b(const double x, const double k) const;
    double get_v_cdm(const double x, const double k) const;
    double get_v_b(const double x, const double k) const;
    double get_Phi(const double x, const double k) const;
    double get_Psi(const double x, const double k) const;
    double get_Pi(const double x, const double k) const;
    double get_Theta(const double x, const double k, const int ell) const;
    double get_Theta_p(const double x, const double k, const int ell) const;
    double get_Nu(const double x, const double k, const int ell) const;
    double get_Source_T(const double x, const double k) const;
    double get_Source_E(const double x, const double k) const;
```

After Tight Coupling we must set the IC of the full system

The right hand side of the Full ODE system

The main method here Integrate the perturbations

Main method: calls integrate pert and then (for next milestone) Computes source functions

Function to get the perturbations f(x,k) from the splines you have made

Other things

- **Polarization and neutrinos:** For neutrinos we keep all the multipoles (~8) in both regimes. For polarisation we don't include any in tight coupling (the IC for the full system follow). If you don't include one or both of these then ignore these equations and put the multipoles to zero wherever they occur in other equations

$$\Theta_\ell^P = 0 \qquad \mathcal{N}_\ell = 0$$

- **Truncation of the Boltzmann hierarchy:** Recall that the equation for a given ell depends on the (ell+1)th moment. Thus we need a way to truncate the hierarchy (see Callin or the website). This means there is a special equation for the largest ell-value (thus you need to set 0,1 and ell_max by hand and in between we have a general formula).

$$\begin{aligned}
\Theta_0' &= -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi', \\
\Theta_1' &= \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right], \\
\Theta_\ell' &= \frac{\ell ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell-1} - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell+1} + \tau'\left[\Theta_\ell - \frac{1}{10}\Pi\delta_{\ell,2}\right], \qquad 2 \le \ell < \ell_{\max} \\
\Theta_\ell' &= \frac{ck}{\mathcal{H}}\Theta_{\ell-1} - c\frac{\ell+1}{\mathcal{H}\eta(x)}\Theta_\ell + \tau'\Theta_\ell, \qquad \ell = \ell_{\max}
\end{aligned}$$

- **Initial conditions:** See the website for a full list of the initial conditions. For setting IC after tight coupling we use the solution from tight coupling and for the quantities we didn't solve for in tight coupling but need for the full system we set these from the other multipoles, e.g. the IC says that so we use this to set the IC for Theta2 from the value of Theta1 and (x, k).

$$\Theta_2 = -\frac{20ck}{45\mathcal{H}}\Theta_1$$

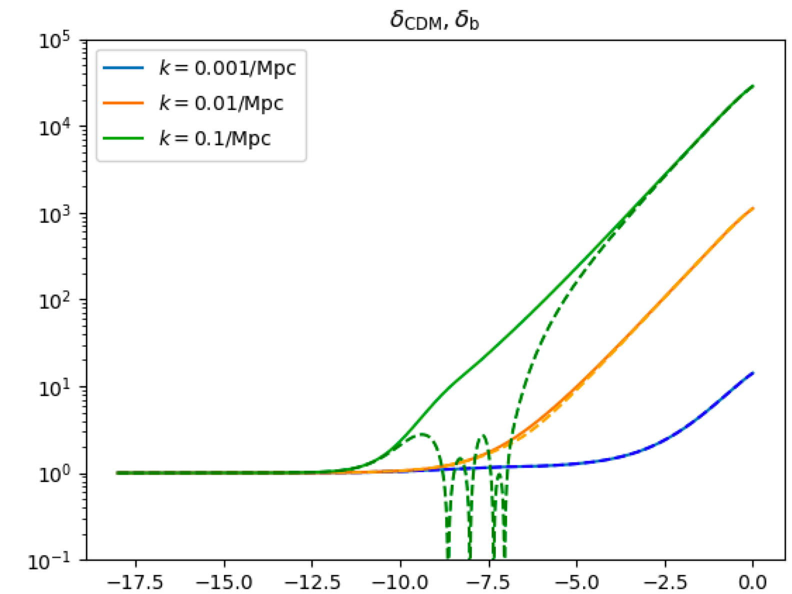
Good luck

- What you are to do is very straight forward on paper… but its still a bit of a mess. A lot of equations that has to be set correctly and two different regimes so there are many things can go wrong when coding this up.

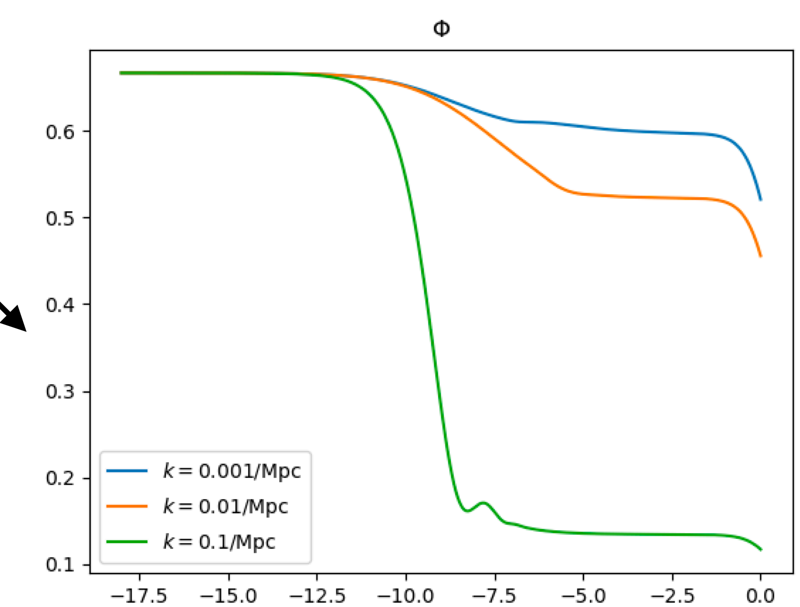
- Therefore **start early!** This milestone is going to to take a lot of time to get correct so you don't want to have to deal with this the last week.

- In this milestone the physics enters much more than in the previous ones. You are expected to be able to describe and explain the results you got based on the physics we have been (and are going to go) through. Why do the perturbation evolve the way they do on different scales? Does your results make sense?

- This is something we will focus on over the next few weeks.



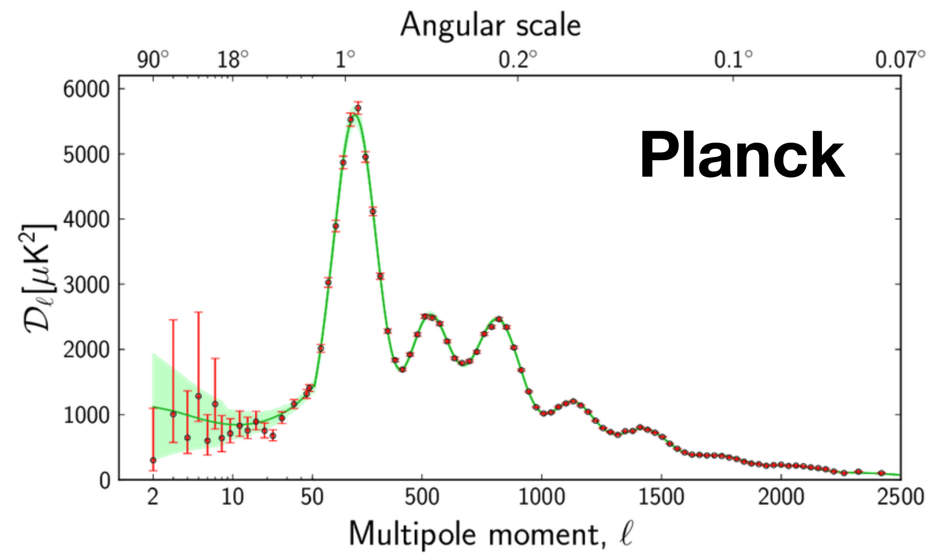
For example: baryon and CDM density.
Why do we see oscillations here?

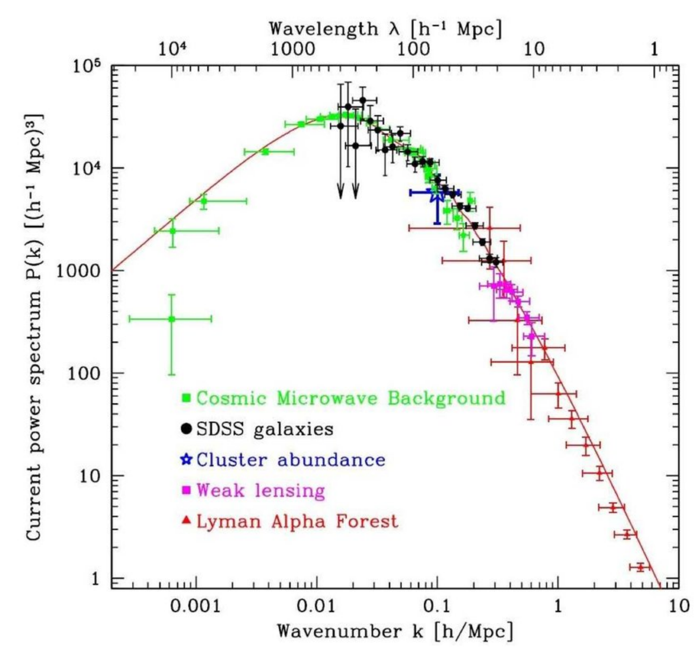


For example: metric potential.
Why does it decay at a certain time?
What determines this?

The next Milestone

$$\vec{Y} = \begin{pmatrix} \Theta_0 \\ \Theta_1 \\ \Theta_2 \\ \dots \\ \Theta_{\ell_{\max}} \\ \Phi \\ \delta_b \\ v_b \\ \dots \end{pmatrix}$$





Galaxy surveys

With the perturbations in hand we are basically just an 'integration' or two away from computing theory predictions that can be contrasted with experiments!