AST3220, spring 2020: Project 2

Annoying repetition of the moral speech from project 1

This project consists of a set of tasks, some analytical, some numerical. You should structure your answers as a report with an introduction, methods, results, discussion and conclusion. It is important that you explain how you think, just writing down a bunch of equations with no explanations will not give you a maximum score. I recommend that you write the report using LaTeX. Posting handwritten lecture notes and solutions to problems is a privilege that belongs to the lecturer alone. Your figures should have a clear layout with proper axis labels and units, and with a caption explaining what the figure shows. The figures should be referenced in the main text. You are also required to hand in your source code in a form that can be easily compiled.

The project: WIMP freeze-out

One of the most promising dark matter models involves so-called Weakly Interacting Massive Particles (WIMPs). It goes without saying that little is known about the nature of WIMPs besides that they are, er, WIMPs. What many have found appealing is that they appear in supersymmetric extensions of the Standard Model, and that what are considered "natural" choices of parameters in these extensions lead to the same amount of dark matter as we infer from observations. This is often called "the WIMP miracle", although I doubt it would impress any of the twelve apostles very much.

Analytical part

Important: In this project we will use units where $\hbar = c = G = k_{\mathrm{B}} = 1$.

In the lectures we discussed how we can model the behaviour of a fluid in a cosmological contex, for example how the particle number density evolves as the Universe expands. In the case where the particle in question interacts with other particles, the most important tool is the Boltzmann equation.

In this project we will follow a gas of WIMP particles, initially in thermal and chemical equilibrium with the rest of the Universe, through the process of decoupling. We will assume that the only relevant processes are annihilation and production of WMP particles through the processes

- $p_1 + p_2 \to p_3 + p_4$
- $p_3 + p_4 \to p_1 + p2$

Here, p_1 and p_2 represent a WIMP particle and a WIMP antiparticle, respectively, while p_3 and p_4 denote two generic Standard Model particles (for example two photons, or an electron and a positron). When the temperature is high enough to keep the two processes of annihilation and creation balanced, the dark matter particles are in thermal and chemical equilibrium. In this case the gases of WIMP particles and antiparticles are characterised by Fermi-Dirac phase space distribution (we assume that WIMPs are fermions).

The general form of the Boltzmann equation for the number density of dark matter particles is given by

$$a^{-3}\frac{d(n_1 a^3)}{dt} = -n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left[\frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} - \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} \right]$$

where the quantity $\langle \sigma v \rangle$ is the so-called thermally averaged cross section. The quantity $n_i^{(0)}$ is the number density of particle i when it is in thermal and chemical equilibrium. The same equation holds for the dark matter antiparticles:

$$a^{-3}\frac{d(n_2 a^3)}{dt} = -n_2^{(0)} n_1^{(0)} \langle \sigma v \rangle \left[\frac{n_2 n_1}{n_2^{(0)} n_1^{(0)}} - \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} \right]$$

We will only consider the total number density of WIMP particles and antiparticles, defined as

$$n_\chi = n_1 + n_2.$$

a) Show that, if we assume the Standard Model particles p_3 and p_4 are always in thermal and chemical equilbrium, the Boltzmann equation can be simplified to

$$\frac{dn_1}{dt} + 3Hn_1 = -\langle \sigma v \rangle \left[n_1 n_2 - n_1^{(0)} n_2^{(0)} \right].$$

Again, the same equation is valid for the dark matter antiparticles. Furthermore, in this project we take the chemical potential for the WIMPs and the anti-WIMPs to be zero, and we can then consider only one Boltzmann equation of the total number density:

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle\sigma v\rangle \left[n_\chi^2 - \left(n_\chi^{(0)}\right)^2\right].$$

b) What are the consequences of equal and vanishing chemical potentials for the distribution functions and number densities of particles and antiparticles? What is the relation between n_χ and n_1 in this case?

We want to follow the WIMPs through the process of decoupling, so we need to solve the Boltzmann equation numerically. For numerical treatment it is convenient to rewrite the Boltzmann equation in dimensionless form. We will use a new time variable

$$x = \frac{m_\chi}{T},$$

where T is the temperature and m_χ is the mass of the WIMP. Also, we introduce the quantity Y defined as

$$Y = \frac{n_\chi}{s},$$

whee s is the entropy density of the Universe. We found an expression for s in the lectures, and in the units we use here it is given by

$$s = \frac{2\pi^2}{45} g_{*s} T^3,$$

where g_{*s} is the effective number of relativistic degrees of freedom contributing to the entropy density. Although it is a function of temperature in general, we will take it to be constant.

What we want to do is to rewrite the Boltzmann equation in terms of Y and x, which are both dimensionless in the system of units we have chosen. We start with Y and note that its time derivative is

$$\frac{dY}{dt} = \frac{d}{dt}\left(\frac{n_\chi}{s}\right) = \frac{1}{s^2}\left(s\frac{dn_\chi}{dt} - n_\chi\frac{ds}{dt}\right).$$

c) By using conservation of entropy, show that

$$\frac{dY}{dt} = -s\langle\sigma v\rangle(Y^2 - Y_{\rm eq}^2).$$

d) Repeated use of the chain rule gives

$$\frac{dY}{dt} = \frac{dY}{dx}\frac{dx}{dT}\frac{dT}{ds}\frac{ds}{dt}.$$

Use this to show that we can write the Boltzmann equation as

$$\frac{dY}{dx} = -\frac{s\langle\sigma v\rangle}{xH(x)}(Y^2 - Y_{\rm eq}^2),$$

where $H(x)$ is the Hubble parameter expressed as a function of x.

We have assumed that the decoupling process takes place in the radiation dominated era and that the first Friedmann equation only involves the energy density of relativistic particles:

$$H^2 = \frac{8\pi}{3}\rho_r,$$

in our units.

e) Show that

$$\frac{s}{H(x)} = \frac{2\pi\sqrt{90}}{45}\frac{g_{*s}}{\sqrt{g_*}}\frac{m_\chi}{\sqrt{8\pi}}\frac{1}{x}.$$

By taking out the factor $1/x$ and multiplying by $\langle\sigma v\rangle$, we define

$$\lambda = \frac{2\pi\sqrt{90}}{45}\frac{g_{*s}}{\sqrt{g_*}}\frac{m_\chi}{\sqrt{8\pi}}\langle\sigma v\rangle.$$

Now we can carry out the last transformation which will bring the Boltzmann equation over to a practical form. It turns out that Y will vary over several orders of magnitude, so it is more convenient to solve for $W = \ln(\lambda Y)$. We set

$$y = \lambda Y$$

and we define the equilibrium value y_{eq} as

$$y_{eq} = 9.35 \times 10^9 \frac{g}{2}\sqrt{\frac{100}{g_*}} \left(\frac{m_\chi}{1000 \text{ GeV}}\right) \left(\frac{\langle\sigma v\rangle}{10^{-10} \text{ GeV}^{-2}}\right) x^{3/2} e^{-x},$$

where g is the number of internal degrees of freedom of the WIMP particle. With these definitions we arrive at the version of the Boltzmann equation we will work with in the following:

$$\frac{dW}{dx} = \frac{1}{x^2}(e^{2W_{eq}-W} - e^W).$$

Numerical part

Finally, we are ready to solve the Boltzmann equation numerically. We can see now that the only parameters in our model is the WIMP mass m_χ and the thermally averaged cross section $\langle\sigma v\rangle$.

f) Write a code to integrate the Boltzmann equation between $1 \leq x \leq 10^3$ and use it to study the decoupling of a WIMP particle of mass $m_\chi =$ 1000 Gev, assuming three different thermally averaged cross sections: $\langle\sigma v\rangle_1 = 10^{-9}$ GeV^{-2}, $\langle\sigma v\rangle_2 = 10^{-10}$ GeV^{-2}, and $\langle\sigma v\rangle_3 = 10^{-11}$ GeV^{-2}. Assume in all cases that the spin of the WIMP particle is 1/2, and that the dark matter particles and antiparticles were initially in thermal and chemical equilibrium with the primordial plasma. For each of the three cases, plot the function y together with the equilibrium function y_{eq}.

g) The abundance of dark matter today is often quantified by

$$\Omega_{\text{dm},0}h^2 = 1.69 \frac{x_f}{20}\sqrt{\frac{100}{g_*}} \left(\frac{10^{-10} \text{ Gev}^{-2}}{\langle\sigma v\rangle}\right),$$

where x_f is the "time" of decoupling, which we take to be the moment when y dropped to 10% of its initial value. Observations give

$$\Omega_{\text{dm},0}h^2 \approx 0.12.$$

Which thermally averaged cross section gives a result closest to the observations?

h) Write a routine which samples the thermally averaged cross section in the range $10^{-14}\ \text{GeV}^{-2} \leq \langle \sigma v \rangle \leq 10^{-7}\ \text{GeV}^{-2}$ and find the interval of cross sections which are consistent with the observed dark matter abundance being in the range $\Omega_{\text{dm},0} h^2 = 0.12 \pm 0.05$

Bonus question: Why is it a bad idea to use an Euler ODE solver in this problem?